Engineering Journal issn: 0125-8281

Article

Stability of Control Systems with Multiple Sector-Bounded Nonlinearities for Inputs Having Bounded Magnitude and Bounded Slope

Suchin Arunsawatwong[a,*] **and Tadchanon Chuman**[b]

Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University, Bangkok 10330, Thailand
E-mail: [a,*]suchin.a@chula.ac.th (Corresponding author), [b]tadchanon01@hotmail.com

Abstract. This paper considers the input-output stability of a control system that is composed of a linear time-invariant multivariable system interconnecting with multiple decoupled time-invariant memoryless nonlinearities. The objectives of the paper are twofold. First and foremost, we prove (under certain assumptions) that if the multivariable Popov criterion is satisfied, then the system outputs and the nonlinearity inputs are bounded for any exogeneous input having bounded magnitude and bounded slope, and for all the nonlinearities lying in given sector bounds. As a consequence of using the convolution algebra, the obtained result is valid for rational and nonrational transfer functions. Second, for the case in which the transfer functions associated with the Popov criterion are rational functions, we develop a useful inequality for stabilizing the system by numerical methods. This is achieved by means of the positive real lemma and known results on linear matrix inequalities. To illustrate the usefulness of the inequality, a numerical example is provided.

Keywords: Input-output stability, sector-bounded nonlinearity, Popov criterion, linear matrix inequalities, numerical stabilization, method of inequalities.

ENGINEERING JOURNAL Volume 27 Issue 5
Received 29 January 2023
Accepted 20 May 2023
Published 31 May 2023
Online at https://engj.org/
DOI:10.4186/ej.2023.27.5.51

DOI:10.4186/ej.2023.27.5.51

1. Introduction

One of the main objectives in control systems design is to ensure that the outputs (or responses) always stay within their prescribed bounds for all inputs that happen or are likely to happen in practice ([1, 2]). For the purpose of design, such inputs are often modelled as time-functions that are restricted in magnitude and slope. During the design procedure, designers have to make sure that the outputs must be bounded for all the inputs, which gives rise to the need for input-output stability investigation. For detailed discussion, see [3, 1, 2].

In this paper, we investigate the input-output stability of a general control system that is an interconnection of linear time-invariant (LTI) subsystems and multiple decoupled sector-bounded nonlinearities in connection with the input set

$$\mathcal{F}_\infty \triangleq \{f : f \in \mathbb{L}^\infty \text{ and } \dot{f} \in \mathbb{L}^\infty\}. \quad (1)$$

The system is depicted in Fig. 1 where $\mathbf{p} \in \mathbb{R}^N$ is a vector of design parameters, $\Psi \triangleq [\psi_1, \psi_2, \ldots, \psi_n]^T$ is the vector of sector-bounded nonlinearities, $\mathbf{z} \triangleq [z_1, z_2, \ldots, z_m]^T$ is the vector of outputs of interest, $\mathbf{v} \triangleq [v_1, v_2, \ldots, v_n]^T$ is the vector of the nonlinearity outputs, $\mathbf{u} \triangleq [u_1, u_2, \ldots, u_n]^T$ is the vectors of the nonlinearity inputs, f denotes an exogenous input that belongs to the set $\mathcal{F}_\infty$. As usual, for a function $x : \mathbb{R}_+ \to \mathbb{R}$, $\|x\|_\infty \triangleq \sup_{t\geq 0} |x(t)|$ and $\mathbb{L}^\infty \triangleq \{x : \|x\|_\infty < \infty\}$.

A noteworthy feature of the set $\mathcal{F}_\infty$ is that when positive bounds M and D are specified, the set

$$\mathcal{F} \triangleq \{f \in \mathcal{F}_\infty : \|f\|_\infty \leq M \quad \text{and} \quad \|\dot{f}\|_\infty \leq D\} \quad (2)$$

is suitable for characterizing inputs that vary persistently for all time, called *persistent inputs*. When all inputs are persistent and do not have stepwise discontinuities, using $\mathcal{F}$ makes the formulation more realistic and more appropriate than using $\mathbb{L}^\infty$; see [3, 1, 2] for details. For different characterizations of the input set and their implications, readers are referred to, e.g., [3, 1, 2, 4, 5].

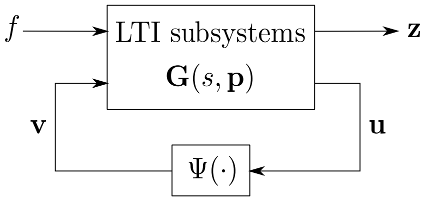

Fig. 1. General configuration for nonlinear control systems considered in the paper.

It may be noted that in connection with the input set $\mathcal{F}$, many researchers (e.g., [6, 7, 3, 1, 2, 8, 9, 10, 5] and also the references therein) have been prompted to develop design methods for linear control systems so that the outputs of interest stay within the bounds for all inputs belonging to the set $\mathcal{F}$. The methods have been applied to solve practical applications (e.g., [11, 12, 13, 14, 15]).

As a special case of this work, the input-output stability property of the unity feedback control system shown in Fig. 2 where $G(s)$ is a strictly proper stable transfer function and ψ is a sector-bounded nonlinearity has been investigated by [16, 17, 18] in connection with the set $\mathcal{F}_\infty$. It has been shown ([16, 17]) that the satisfaction of the well-known Popov criterion (e.g., [19]) implies that the outputs v_1 and v_2 are bounded for all $f \in \mathcal{F}_\infty$ and for any nonlinearity ψ lying in a sector bound. It should be noted however that in the original work, the Popov criterion has been used for investigating the asymptotic stability property of nonlinear systems.

It may be noted further that for another input set

$$\mathcal{F}_2 \triangleq \{f : f \in \mathbb{L}^2 \text{ and } \dot{f} \in \mathbb{L}^2\} \quad (3)$$

where $\mathbb{L}^2 \triangleq \{x : \int_0^\infty |x(t)|^2 dt < \infty\}$, Desoer [20] proves that if the Popov criterion is satisfied, then the outputs v_1 and v_2 of the system in Fig. 2 are bounded for any $f \in \mathcal{F}_2$ and for any ψ lying in a sector bound. The extension of this result to the case of multiple nonlinearities is given in [21] and can be obtained by using the passivity properties.

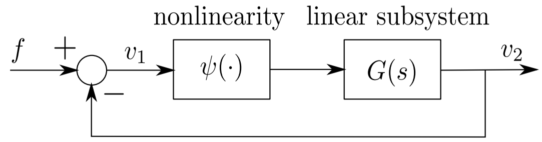

Fig. 2. Feedback control system with a nonlinearity.

The objectives of this paper are twofold. First and foremost, we prove that for the control system in Fig. 1, the satisfaction of the multivariable Popov criterion implies that the output vector $\mathbf{z}$ and the nonlinearity input vector $\mathbf{u}$ are bounded for any input $f \in \mathcal{F}_\infty$, which is a generalization of the stability result presented in [17]. This result hold for the case in which the LTI subsystems can be either rational or nonrational transfer functions. Second, for the case in which the LTI subsystems are rational transfer functions, we develop a condition for ensuring the input-output stability of the system in the form of a readily computable inequality; this is achieved by means of the positive real lemma and known results on linear matrix inequalities. As a consequence, such a condition provides a practical inequality for determining a stabilizing controller by numerical methods.

The organization of the article is as follows. Section 2 provides a detailed description of the system in Fig. 1. Section 3 presents the mathematical results on the stability of the system in Fig. 1, which is the main contribution of this work. Based on the obtained stability results, Section 4 develops a practical inequality that is suitable for numerical stabilization. To illustrate the usefulness of the obtained inequality, a stabilizing controller for a system with two nonlinearities is designed in Section 5. Finally, conclusions are given in Section 6.

2. System Description

Suppose that the LTI subsystems are represented by a transfer matrix $\mathbf{G}(s,\mathbf{p})$ described by

$$\mathbf{G}(s,\mathbf{p}) = \begin{bmatrix} \mathbf{G}_{zf}(s,\mathbf{p}) & \mathbf{G}_{zv}(s,\mathbf{p}) \\ \mathbf{G}_{uf}(s,\mathbf{p}) & \mathbf{G}_{uv}(s,\mathbf{p}) \end{bmatrix},$$

where $\mathbf{G}_{zf}(s,\mathbf{p}) \triangleq [G_{z_if}(s,\mathbf{p})]_{m\times 1}$, $\mathbf{G}_{zv}(s,\mathbf{p}) \triangleq [G_{z_iv_j}(s,\mathbf{p})]_{m\times n}$, $\mathbf{G}_{uf}(s,\mathbf{p}) \triangleq [G_{u_if}(s,\mathbf{p})]_{n\times 1}$, $\mathbf{G}_{uv}(s,\mathbf{p}) \triangleq [G_{u_iv_j}(s,\mathbf{p})]_{n\times n}$. Then the mathematical model of the system in Fig. 1 is described by

$$\begin{aligned} & z_i = g_{z_if} * f + \sum_{j=1}^{n}(g_{z_iv_j} * v_j), \quad i = 1,2,\ldots,m \\ & \left.\begin{aligned} u_i &= g_{u_if} * f + \sum_{j=1}^{n}(g_{u_iv_j} * v_j) \\ v_i &= \psi_i(u_i) \end{aligned}\right\}, \; i = 1,2,\ldots,n \end{aligned} \tag{4}$$

where g_{z_if}, $g_{z_iv_j}$, g_{u_if} and $g_{u_iv_j}$ denote the inverse Laplace transforms of $G_{z_if}(s,\mathbf{p})$, $G_{z_iv_j}(s,\mathbf{p})$, $G_{u_if}(s,\mathbf{p})$, and $G_{u_iv_j}(s,\mathbf{p})$, respectively. As usual the symbol $*$ denotes the convolution; i.e., for $x : \mathbb{R}_+ \to \mathbb{R}$ and $y : \mathbb{R}_+ \to \mathbb{R}$,

$$(x * y)(t) = \int_0^t x(t-\tau)y(\tau)d\tau, \quad t > 0.$$

The definition of sector condition is introduced as follows. For $i = 1,2,\ldots,n$, the nonlinearity $\psi_i \in$ sector$[0,k_i]$ if

$$\psi_i(0) = 0 \quad \text{and} \quad 0 \le \frac{\psi_i(\sigma)}{\sigma} \le k_i \quad \forall\, \sigma \ne 0. \tag{5}$$

See Fig. 3 for the graphical description.

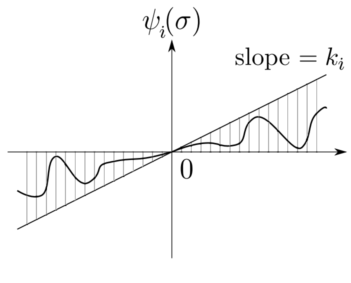

Fig. 3. Nonlinearity $\psi_i \in$ sector$[0,k_i]$.

Assumption 1. *The nonlinearities $\psi_i : \mathbb{R} \to \mathbb{R}$ ($i = 1,2,\ldots,n$) are functions that are piecewise continuous and time-invariant and satisfy conditions* (5).

Assumption 2. *The linear part of the system* (4) *is a time-invariant and non-anticipative system with zero initial conditions.*

In order to make this paper's contribution applicable to the case of rational and non-rational transfer functions, the following notation is useful. Let $\mathcal{A}$ denote the convolution algebra whose elements take the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=0}^{\infty} g_i\delta(t-t_i), & t \ge 0 \\ 0, & t < 0 \end{cases} \tag{6}$$

where $\delta(\cdot)$ is the Dirac delta function, $0 = t_0 < t_1 < t_2 < \cdots$ are constants,

$$\int_0^{\infty} |g_a(t)| < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} |g_i| < \infty.$$

Note that all elements of $\mathcal{A}$ are the impulse responses of bounded-input bounded-output (BIBO) stable transfer functions. For the details on the algebra $\mathcal{A}$, see, e.g., [21].

Assumption 3. *For $i = 1,2,\ldots,m$, $g_{z_if} \in \mathcal{A}$. For $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$, $g_{z_iv_j} \in \mathcal{A}$. For $i = 1,2,\ldots,n$, $g_{u_if} \in \mathcal{A}$. For $i,j = 1,2\ldots,n$, $g_{u_iv_j} \in \mathcal{A}$, $\dot{g}_{u_iv_j} \in \mathcal{A}$ and there exists a sufficiently small $\alpha > 0$ such that*

$$\int_0^{\infty} e^{2\alpha t} g_{u_iv_j}^2(t)dt < \infty. \tag{7}$$

3. Stability Conditions

This section derives the main theoretical result of the article. The result is presented in Theorem 3.5, providing a stability condition for the system (4) in connection with the set $\mathcal{F}_\infty$.

Define the space $\mathbb{L}_n^\infty \triangleq \underbrace{\mathbb{L}^\infty \times \mathbb{L}^\infty \ldots \times \mathbb{L}^\infty}_{n}$. The definitions of input-output stability used in the paper are given as follows.

Definition 3.1. The system (4) is said to be *input-output stable* if $\mathbf{z} \in \mathbb{L}_m^\infty$ and $\mathbf{u} \in \mathbb{L}_n^\infty$ for any $f \in \mathcal{F}_\infty$.

Definition 3.2. The system (4) is said to be *absolutely input-output stable* if it is input-output stable for all $\psi_i \in$ sector$[0,k_i]$ $(i = 1,2,\ldots,n)$.

The following two lemmas are used to prove Theorem 3.5. Lemma 3.3 is a generalization of a well-known lemma given in [19] whereas Lemma 3.4 is a generalization of the main lemma given in [17].

Lemma 3.3. *Let $y_i, w_i, x_i \in \mathbb{L}^2$ for $i = 1, 2, \ldots, n$. Define*

$$\begin{aligned} \mathbf{Y}(j\omega) &\triangleq \left[Y_1(j\omega), Y_2(j\omega), \ldots, Y_n(j\omega)\right]^T, \\ \mathbf{W}(j\omega) &\triangleq \left[W_1(j\omega), W_2(j\omega), \ldots, W_n(j\omega)\right]^T, \\ \mathbf{X}(j\omega) &\triangleq \left[X_1(j\omega), X_2(j\omega), \ldots, X_n(j\omega)\right]^T \end{aligned}$$

where $Y_i(j\omega), W_i(j\omega)$ and $X_i(j\omega)$ denote the Fourier transforms of y_i, w_i and x_i, respectively. If

$$\mathbf{Y}(j\omega) = \mathbf{H}(j\omega)\mathbf{X}(j\omega) + \mathbf{W}(j\omega) \tag{8}$$

where

$$\frac{\mathbf{H}(j\omega) + \mathbf{H}^*(j\omega)}{2} \geq \delta I > 0, \quad \forall\, \omega \geq 0, \tag{9}$$

then

$$\sum_{i=1}^{n} \int_0^\infty y_i(t)x_i(t)dt + \frac{1}{4\delta}\sum_{i=1}^{n}\int_0^\infty w_i^2(t)dt \geq 0.$$

Proof. See Appendix A. □

Define

$$\mathbf{H}(s, \mathbf{p}) \triangleq -(I + sQ)\mathbf{G}_{uv}(s, \mathbf{p}) + K \tag{10}$$

where $Q \triangleq \operatorname{diag}(q_1, q_2, \ldots, q_n)$ with $q_i \in \mathbb{R}$ for all i, and $K \triangleq \operatorname{diag}(1/k_1, 1/k_2, \ldots, 1/k_n)$ with $k_i > 0$ for all i. Also, define

$$r_i \triangleq g_{u_if} * f, \quad i = 1, 2, \ldots, n. \tag{11}$$

Lemma 3.4. *Consider the system (4) and let Assumptions 1–3 hold. Let $f \in \mathcal{F}_\infty$ and let $\psi_i \in \operatorname{sector}[\varepsilon, k_i - \varepsilon]$ for all i where $\varepsilon > 0$ is arbitrarily small. If there exist $\beta > 0$ and $q_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$) satisfying*

$$\frac{\mathbf{H}(j\omega) + \mathbf{H}^*(j\omega)}{2} \geq \beta I, \quad \forall\, \omega \geq 0, \tag{12}$$

then the following inequality holds for all $t \geq 0$.

$$\begin{aligned} &\sum_{i=1}^{n}\int_0^t e^{2\alpha t}v_i^2(\tau)d\tau \leq \\ &\qquad \sum_{i=1}^{n}\int_0^t \frac{e^{2\alpha\tau}}{\beta^2}[r_i(\tau) + q_i\dot{r}_i(\tau)]^2 d\tau \\ &\qquad + \sum_{i=1}^{n}\frac{2q_i}{\beta}\int_0^{u_i(0)}\psi_i(u_i)du_i. \end{aligned}$$

Proof. See Appendix A. □

We are now ready to state the main result of the paper.

Theorem 3.5. *Consider the system (4) where k_i ($i = 1, 2, \ldots, n$) are given, and let Assumptions 1–3 hold. Then the system (4) is absolutely input-output stable if there exist $\beta > 0$ and $q_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$) such that (12) is satisfied.*

Proof. According to [19] and [17], Theorem 3.5 will be proved for the nonlinearity ψ_i in the reduced sector $[\varepsilon, k_i - \varepsilon]$ ($i = 1, 2, \ldots, n$) where $\varepsilon > 0$ is arbitrarily small.

From (4) and (11), it follows that for each i,

$$u_i(t) = r_i(t) + \sum_{j=1}^{n}\int_0^t g_{u_iv_j}(t - \tau)v_j(t)d\tau. \tag{13}$$

After introducing the factors $e^{-\alpha(t-\tau)}$ and $-e^{-\alpha(t-\tau)}$ in (13) and applying the triangular and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} |u_i(t)| \leq |r_i(t)| + \sum_{j=1}^{n}\Bigg[&\left(\int_0^t e^{2\alpha\lambda}g_{u_iv_j}^2(\lambda)d\lambda\right)^{1/2} \\ &\cdot e^{-\alpha t}\left(\textstyle\int_0^t e^{2\alpha\tau}v_j^2(\tau)d\tau\right)^{1/2}\Bigg]. \end{aligned} \tag{14}$$

By using Lemma 3.4, condition (12) implies that the following inequality holds for a sufficiently small $\alpha > 0$.

$$\begin{aligned} &\sum_{j=1}^{n}\int_0^\infty e^{2\alpha t}v_j^2(\tau)d\tau \leq \\ &\quad \sum_{j=1}^{n}\int_0^t \frac{e^{2\alpha\tau}}{\beta^2}\Big(r_j(\tau) + q_j\dot{r}_j(\tau)\Big)^2 d\tau \\ &\quad + \sum_{j=1}^{n}\frac{2q_j}{\beta}\int_0^{u_j(0)}\psi_j(u_j)du_j, \quad \forall\, t \geq 0. \end{aligned} \tag{15}$$

Since $\int_0^\infty e^{2\alpha t}v_i^2(\tau)d\tau \leq \sum_{j=1}^{n}\int_0^\infty e^{2\alpha t}v_j^2(\tau)d\tau$ for each i, it follows from (14) and (15) that

$$\begin{aligned} |u_i(t)| \leq |r_i(t)| &\\ +\sum_{j=1}^{n}\Bigg[&\left(\int_0^t e^{2\alpha\lambda}g_{u_iv_j}^2(\lambda)d\lambda\right)^{1/2} \\ &\cdot\left(\frac{1}{\beta^2}\sum_{k=1}^{n}\int_0^t e^{-2\alpha(t-\tau)}\big(r_k(\tau) + q\dot{r}_k(\tau)\big)^2 d\tau\right. \\ &\left. +\sum_{k=1}^{n}\frac{2q_k}{\beta}e^{-2\alpha t}\int_0^{u_k(0)}\psi_k(u_k)du_k\right)^{1/2}\Bigg]. \end{aligned} \tag{16}$$

Let $f \in \mathcal{F}_\infty$. Then it follows from (11) that for each i, the condition $g_{u_if} \in \mathcal{A}$ implies that $r_i \in \mathcal{F}_\infty$. By using (7), we see that for each i, the right-hand side of (16) is finite for all $t \geq 0$. Thus, $\mathbf{u} \in \mathbb{L}_n^\infty$ for any $f \in \mathcal{F}_\infty$.

By the continuity of Ψ, $\mathbf{u} \in \mathbb{L}_n^\infty$ implies $\mathbf{v} \in \mathbb{L}_n^\infty$. Hence, it follows from (4) that the conditions $g_{z_if} \in \mathcal{A}$ for all i and $g_{z_iv_j} \in \mathcal{A}$ for all i, j imply that $\mathbf{z} \in \mathbb{L}_m^\infty$ for any $f \in \mathcal{F}_\infty$. Therefore, the proof is complete by Definition 3.2. □

When the system (4) has only one sector-bounded nonlinearity (that is, $n = 1$), condition (12) becomes

$$-\mathrm{Re}(1+j\omega q)G_{uv}(j\omega, \mathbf{p})+\frac{1}{k} \geq \beta > 0 \quad \forall\omega \geq 0, \quad (17)$$

which is the well-known Popov criterion. Evidently, condition (12) can be seen as a *multivariable* version of (17).

The criterion (17) results in a useful graphical test for the absolute input-output stability of control systems with one sector-bounded nonlinearity. Further, based on (17), Mai, Arunsawatwong and Abed [22] develop a procedure for *numerical stabilization* (that is to say, determining, by using numerical methods, a vector $\mathbf{p} \in \mathbb{R}^N$ for which the system is absolutely input-output stable). By contrast, when the system has multiple sector-bounded nonlinearities, the absolute input-output stability test for the system (4) with the criterion (12) in general becomes more complicated. Furthermore, we find it difficult to develop a procedure for numerical stabilization by direct use of (12).

4. Numerical Stabilization

In this section, attention is restricted only to the case in which all elements of the transfer matrix $\mathbf{G}_{uv}(s, \mathbf{p})$ are rational functions. Then, by using Theorem 3.5, we develop a useful inequality that can be used to compute a design parameter vector $\mathbf{p} \in \mathbb{R}^N$ for which the system (4) is absolutely input-output stable. To this end, we make the following assumption.

Assumption 4. *The transfer functions* $G_{u_i v_j}(s, \mathbf{p})$ *are strictly proper rational functions for all* $i, j = 1, 2, \ldots, n$.

The key tool to be used here is the positive real lemma ([23, 24]), which is stated as follows.

Lemma 4.1 ([23, 24])**.** *Let* $\mathbf{X}(s)$ *be an* $n \times n$ *transfer matrix with a state space realization*

$$\mathbf{X}(s) \sim \left[\begin{array}{c|c} A_X & B_X \\ \hline C_X & D_X \end{array}\right].$$

Then

$$\mathbf{X}(j\omega) + \mathbf{X}^*(j\omega) > 0, \quad \forall\,\omega \in \mathbb{R} \quad (18)$$

if and only if there exists $P = P^T$ *such that*

$$\begin{bmatrix} A_X^T P + PA_X & PB_X - C_X^T \\ B_X^T P - C_X & -D_X - D_X^T \end{bmatrix} < 0. \quad (19)$$

In connection with Assumption 4, let a state space realization of $\mathbf{G}_{uv}(s, \mathbf{p})$ be given by

$$\mathbf{G}_{uv}(s, \mathbf{p}) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]. \quad (20)$$

Then it follows immediately from (10) that a state space realization of the transfer matrix $\mathbf{H}(s, \mathbf{p})$ is given by

$$\mathbf{H}(s, \mathbf{p}) \sim \left[\begin{array}{c|c} A & B \\ \hline -C - QCA & -QCB + K \end{array}\right].$$

By applying Lemma 4.1 to the transfer matrix $\mathbf{H}(s, \mathbf{p})$, it is easy to see that for any $q_i \in \mathbb{R}$ ($i = 1, 2, \ldots, n$), there exists $\beta > 0$ such that criterion (12) is satisfied if and only if there exists $P = P^T$ such that

$$L < 0, \quad (21)$$

$$L \triangleq \begin{bmatrix} A^T P + PA & PB + C^T + A^T C^T Q \\ B^T P + C + QCA & QCB + B^T C^T Q - 2K \end{bmatrix}.$$

Now consider the following LMI problem:

$$\begin{array}{cc} \min & \lambda \\ \text{subject to} & L < \lambda I \end{array} \quad (22)$$

where $P = P^T$ and Q are the optimization variables. Let λ^* denote the minimum value of λ in problem (22). Thus it readily follows that there exists $P = P^T$ and Q satisfying (21) if and only if

$$\lambda^* < 0. \quad (23)$$

Hence, the above discussion is formally stated as follows.

Proposition 4.2. *Consider the system* (4) *where* k_i *(*$i = 1, 2, \ldots, n$*) are given, and let Assumptions 1–4 hold. Let a state space realization of* $\mathbf{G}_{uv}(s, \mathbf{p})$ *be given by* (20)*. Then the system* (4) *is absolutely input-output stable if inequality* (23) *is satisfied.*

Proof. The proof readily follows from Theorem 3.5 and the above discussion. □

Problem (22), which is called the generalized eigenvalue minimization problem, can be solved efficiently by using convex optimization methods. Consequently, the number λ^* is readily obtainable in practice and inequality (23) is more computationally tractable than the multivariable Popov criterion (12). In this work, the LMI Control Toolbox for MATLAB ([25]) is employed.

In connection with the system (4), it is easy to see that λ^* is a function of $\mathbf{p}$. Since $\lambda^*(\mathbf{p})$ is finite for all $\mathbf{p} \in \mathbb{R}^N$, it follows that inequality (23) is suitable for solution by numerical methods ([26]). In order to use inequality (23) in conjunction with the method of inequalities ([27, 2]), we replace (23) with the following inequality

$$\lambda^*(\mathbf{p}) \leq -\varepsilon_0, \quad 0 < \varepsilon_0 \ll 1 \quad (24)$$

where ε_0 is specified by designers. With any vector $\mathbf{p}$ that satisfies inequality (24), the system (4) is guaranteed to be absolutely input-output stable.

DOI:10.4186/ej.2023.27.5.51

In practice, a solution $\mathbf{p}$ of inequality (24) can be obtained by employing a numerical algorithm to search in the space $\mathbb{R}^N$. In this work, a search algorithm called the moving-boundaries-process (MBP) is used because it is simple to implement and still effective. See [2, 27] for the detail of the MBP algorithm. It may be noted that other algorithms for solving inequalities may also be used. For further details, see Chapters 7 and 8 of [2] and the references therein.

It is worth noting at this point that searching for a solution of inequality (24) in the space of design parameter vector $\mathbf{p}$ is in general a non-convex problem. As a result, the algorithm could sometimes be hindered by a computational trap. However, as long as a solution exists, this can be easily overcome in practice, for example, by changing a new starting point which is sufficiently far away from the trap or temporarily relaxing the bound $-\varepsilon_0$ so that the algorithm can escape from the trap. More detailed discussion on this can be found in [2].

As will be demonstrated in Section 5, with an appropriate controller structure, a solution of (24) is usually not difficult to obtain although the convergence of numerical search algorithms depends on starting points.

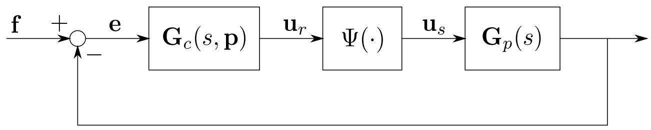

Fig. 4. Nonlinear control system.

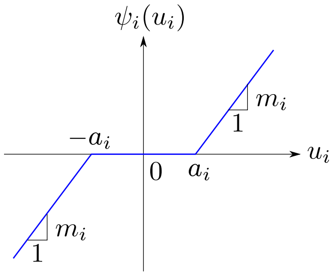

Fig. 5. Dead-zone characteristic of ψ_i.

5. Numerical Example

In this section, we consider the control system shown in Fig. 4 where $\mathbf{f} \triangleq [f_1, f_2]^T$ is the input vector, $\mathbf{G}_p(s)$ is the plant transfer matrix, Ψ is a nonlinearity vector, and $\mathbf{G}_c(s, \mathbf{p})$ is the controller transfer matrix. Also, $\mathbf{e} \triangleq [e_1, e_2]^T$, $\mathbf{u}_r \triangleq [u_{r1}, u_{r2}]^T$, and $\mathbf{u}_r \triangleq [u_{s1}, u_{s2}]^T$ are the error vector, control vector, and plant input vector, respectively.

In the following, a plant model taken from [28] will be used where $\mathbf{G}_p(s)$ is

$$\mathbf{G}_p(s) = \begin{bmatrix} \dfrac{s^2 - 0.2s - 0.1}{s^3 + 2s^2 + s + 1} & \dfrac{-0.1s^2 - 1.2s + 0.1}{s^3 + 2s^2 + s + 1} \\ \dfrac{0.1s^2 - 0.3s - 1}{s^3 + 2s^2 + s + 1} & \dfrac{-s^2 - 2.1s - 0.7}{s^3 + 2s^2 + s + 1} \end{bmatrix}.$$

For the purpose of illustration, let the nonlinearity Ψ be described by

$$\Psi(\mathbf{u}) = [\psi_1(u_1), \psi_2(u_2)]^T$$

where ψ_1 and ψ_2 are dead-zone functions shown in Fig. 5 with the parameters

$$m_1 = m_2 = 1 \quad \text{and} \quad a_1 = a_2 = 0.2.$$

Obviously, the functions ψ_1 and ψ_2 satisfy Assumption 1 where $\psi_1, \psi_2 \in \text{sector}[0, 1]$. Further, let f_1 be an input with bounded magnitude and bounded slope and, for simplicity, let $f_2 = 0$.

It is easy to see that the system in Fig. 4 can be represented as the one in Fig. 1 where $f = f_1$, $\mathbf{z} = \mathbf{e}$, $\mathbf{u} = \mathbf{u}_r$, $\mathbf{v} = \mathbf{u}_s$. The transfer matrices $\mathbf{G}_{zf}(s)$, $\mathbf{G}_{zv}(s)$, $\mathbf{G}_{uf}(s)$ and $\mathbf{G}_{uv}(s)$ are as follows.

$$\begin{aligned} \mathbf{G}_{zf}(s) &= [1, 0]^T, \\ \mathbf{G}_{zv}(s) &= -\mathbf{G}_p(s), \\ \mathbf{G}_{uf}(s, \mathbf{p}) &= \mathbf{G}_c(s, \mathbf{p}), \\ \mathbf{G}_{uv}(s, \mathbf{p}) &= -\mathbf{G}_c(s, \mathbf{p})\mathbf{G}_p(s). \end{aligned}$$

For the case of $\mathbf{G}_c(s, \mathbf{p}) = I$, it is easy to verify that Assumption 3 is satisfied so that Proposition 4.2 can be used. By solving (22), we have $\lambda^* = 0.286$; in this case, the absolute input-output stability of the system cannot be guaranteed by Proposition 4.2. To investigate the system stability for the dead-zone functions, a numerical simulation is carried out where the system is subjected to the test input $f_1^* \in \mathcal{F}_\infty$, shown in Fig. 6. The graphs of z_1, z_2, u_1, u_2 in response to f_1^* are given in Fig. 7. Clearly, all the responses blow up and hence the system is not absolutely input-output stable.

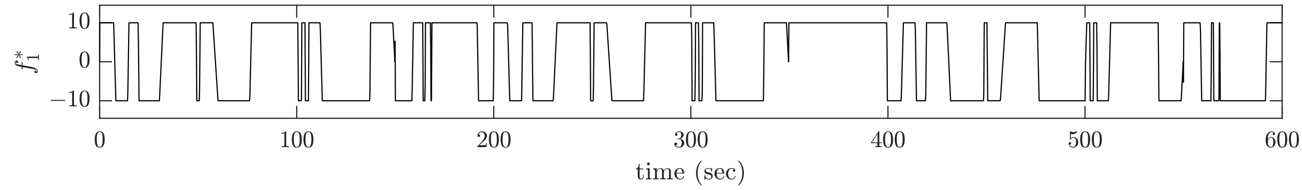

Fig. 6. Waveform of the test input f_1^* where $\|f^*\|_\infty = 10$ and $\|\dot{f}\|_\infty = 100$.

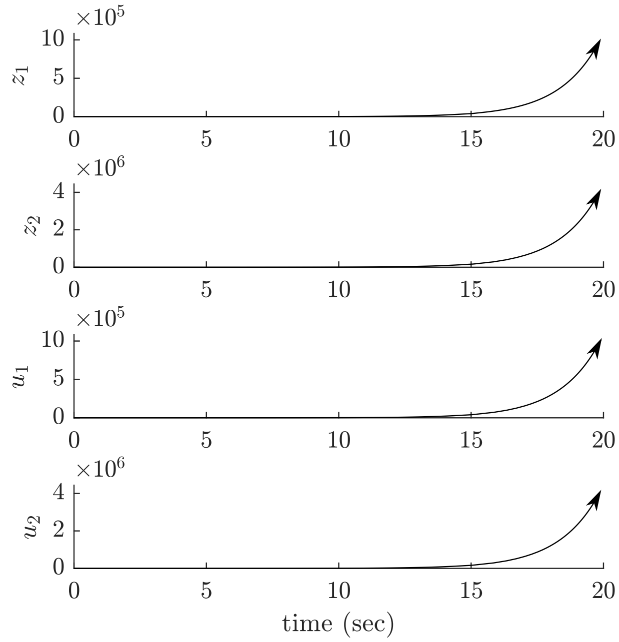

Fig. 7. System responses corresponding to f_1^* for the case in which $\mathbf{G}_c(s, \mathbf{p}) = I$.

Now, we will determine a controller (compensator) $\mathbf{G}_c(s, \mathbf{p})$ so that the system is absolutely input-output stable. To this end, let the controller $\mathbf{G}_c(s, \mathbf{p})$ be of the form

$$\mathbf{G}_c(s, \mathbf{p}) = \begin{bmatrix} p_1 & 0 \\ 0 & \dfrac{p_2(s+p_3)}{s+p_4} \end{bmatrix} \quad (25)$$

where $\mathbf{p} \triangleq [p_1, p_2, p_3, p_4]^T$ satisfies the constraints

$$p_i > 0, \quad i = 1, 2, 3, 4. \quad (26)$$

It is easy to verify that for the system with the controller structure described by (25) and (26), Assumptions 2–4 are always satisfied. Therefore, it follows from Proposition 4.2 that a stabilizing solution $\mathbf{p}$ is obtained by solving the inequality

$$\lambda^*(\mathbf{p}) \le -10^{-6} \quad (27)$$

together with inequalities (26).

The numerical results with four different starting points $\mathbf{p}_0$ are given below where the superscripts I–IV indicate the case numbers; their summary is in Table 1.

Case I. A starting point $\mathbf{p}_0^{\mathrm{I}} = [1, 1, 0, 0]^T$ is used, resulting in $\mathbf{G}_c(s, \mathbf{p}_0^{\mathrm{I}}) = I$. After 200 iterations, the MBP algorithm yields a point $\mathbf{p}^{\mathrm{I}}$ where

$$\mathbf{p}^{\mathrm{I}} = [0.327, 0.765, 0.0916, 0.0916]^T,$$

$$\mathbf{G}_c(s, \mathbf{p}^{\mathrm{I}}) = \begin{bmatrix} 0.327 & 0 \\ 0 & 0.765 \end{bmatrix} \quad (28)$$

and

$$\lambda^*(\mathbf{p}^{\mathrm{I}}) = 0.128. \quad (29)$$

In this case, the MBP algorithm becomes trapped and fails to converge to a solution of inequality (27). Since the vector $\mathbf{p}^{\mathrm{I}}$ does not satisfy (27), the absolute input-output stability of the system cannot be guaranteed by Proposition 4.2.

Case II. A starting point $\mathbf{p}_0^{\mathrm{II}}$ is chosen by setting $\mathbf{p}_0^{\mathrm{II}} = [p_1^{\mathrm{I}}, p_2^{\mathrm{I}}, p_3^{\mathrm{I}}, 1]^T$ where p_1^{I}, p_2^{I} and p_3^{I} are the first, the second, and the third elements of $\mathbf{p}^{\mathrm{I}}$. In this case, the algorithm locates a solution $\mathbf{p}^{\mathrm{II}}$ of inequality (27) in 13 iterations where

$$\mathbf{p}^{\mathrm{II}} = [0.164, 0.612, 0.0870, 2.30]^T,$$

$$\mathbf{G}_c(s, \mathbf{p}^{\mathrm{II}}) = \begin{bmatrix} 0.164 & 0 \\ 0 & \dfrac{0.612(s+0.0870)}{s+2.30} \end{bmatrix} \quad (30)$$

and

$$\lambda^*(\mathbf{p}^{\mathrm{II}}) = -1.94 \times 10^{-2}. \quad (31)$$

Case III. From a starting point $\mathbf{p}_0^{\mathrm{III}} = [1, 2, 3, 2]^T$, the algorithm locates a solution $\mathbf{p}^{\mathrm{III}}$ of inequality (27) in 25 iterations where

$$\mathbf{p}^{\mathrm{III}} = [0.613, 2.05, 2.51, 74.8]^T,$$

$$\mathbf{G}_c(s, \mathbf{p}^{\mathrm{III}}) = \begin{bmatrix} 0.613 & 0 \\ 0 & \dfrac{2.05(s+2.51)}{s+74.8} \end{bmatrix} \quad (32)$$

and

$$\lambda^*(\mathbf{p}^{\mathrm{III}}) = -1.73 \times 10^{-3}. \quad (33)$$

Case IV. From a starting point $\mathbf{p}_0^{\mathrm{IV}} = [0.01, 10, 20, 200]^T$, the algorithm locates a solution $\mathbf{p}^{\mathrm{IV}}$ of inequality (27) in 48 iterations where

$$\mathbf{p}^{\mathrm{IV}} = [0.0236, 21.3, 1.27, 223]^T,$$

case	starting point $\mathbf{p}_0$	$\lambda^*(\mathbf{p}_0)$	stabilizing solution $\mathbf{p}$	number of iterations	$\lambda^*(\mathbf{p})$
I	$[1, 1, 0, 0]^T$	0.286	$[0.327, 0.765, 0.0916, 0.0916]^T$	200	0.128
II	$[0.327, 0.765, 0.0916, 1]^T$	0.300	$[0.164, 0.612, 0.0870, 2.30]^T$	13	-0.0194
III	$[1, 2, 3, 2]^T$	0.0893	$[0.613, 2.05, 2.51, 74.8]^T$	25	-1.73×10^{-3}
IV	$[0.01, 10, 20, 200]^T$	0.0716	$[0.0236, 21.3, 1.27, 223]^T$	48	-0.0316

Table 1. Stabilizing solutions from different starting points.

$$\mathbf{G}_c(s, \mathbf{p}^{\mathrm{IV}}) = \begin{bmatrix} 0.613 & 0 \\ 0 & \dfrac{2.05(s+2.51)}{s+74.8} \end{bmatrix} \quad (34)$$

and

$$\lambda^*(\mathbf{p}^{\mathrm{IV}}) = -3.16 \times 10^{-2}. \quad (35)$$

To verify the results for Cases I–IV, simulations are carried out with the controllers (28), (30), (32) and (34) for the test input f_1^* as before. The corresponding system responses due to f_1^* for Cases I-IV are displayed in Figs. 8–11, respectively.

From Fig. 8, one can see that with the controller (28) obtained in Case I, the system responses blow up and hence the system is not absolutely input-output stable. In Cases II–IV, the MBP algorithm locates solution of inequality (27) quickly. From Figs. 9–11, it can be seen that with the controllers (30), (32) and (34), the system responses are bounded, which agrees with the stability result given by Proposition 4.2.

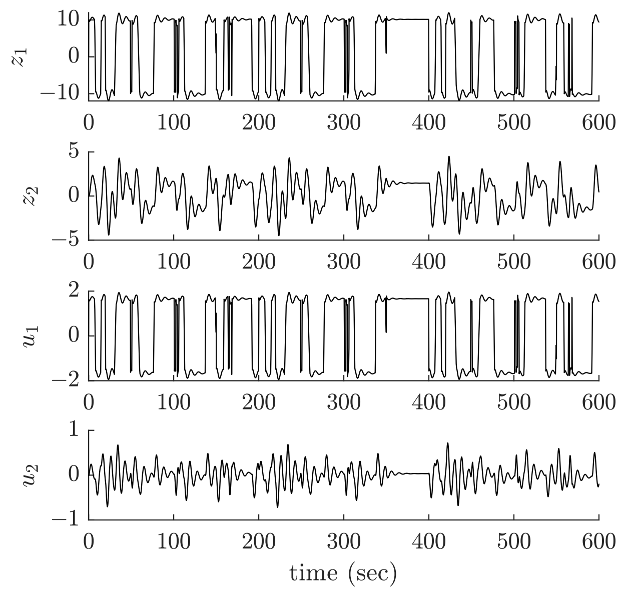

Fig. 9. System responses corresponding to f_1^* with the controller (30).

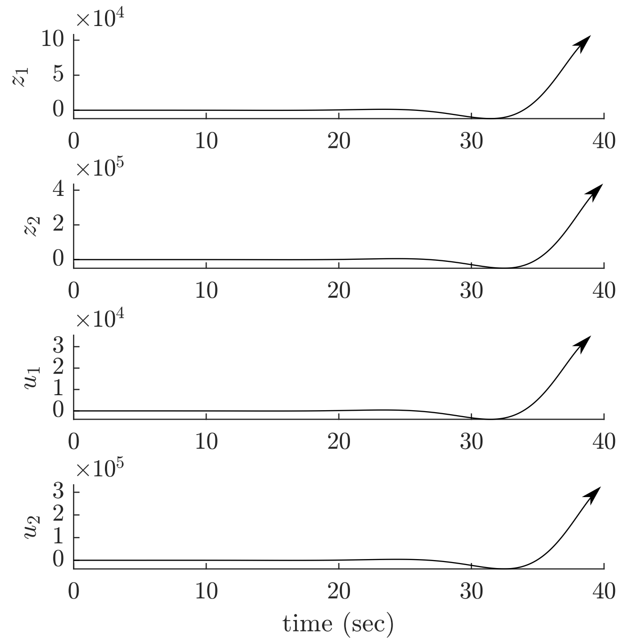

Fig. 8. System responses corresponding to f_1^* with the controller (28).

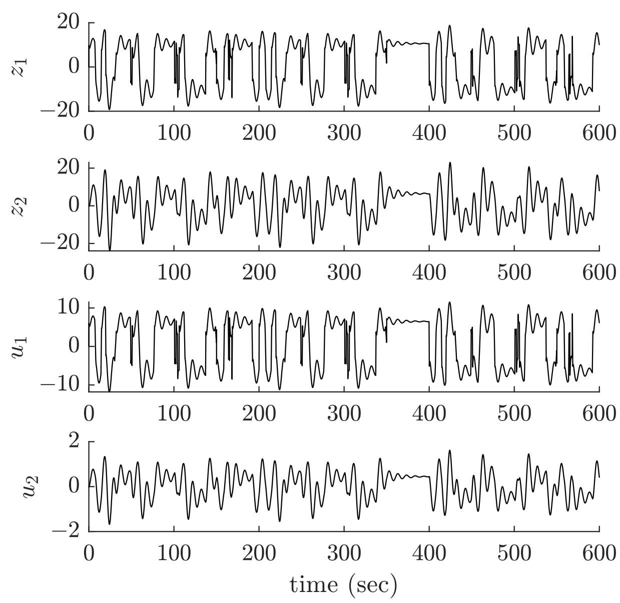

Fig. 10. System responses corresponding to f_1^* with the controller (32).

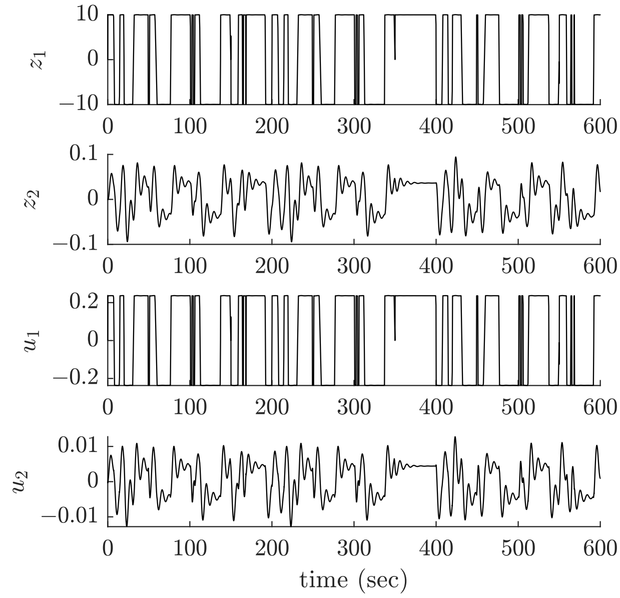

Fig. 11. System responses corresponding to f_1^* with the controller (34).

6. Conclusions

In this paper, we consider the nonlinear system (4) and present the main theoretical result in Theorem 3.5. The result states that if the transfer matrix $\mathbf{G}_{uv}(s, \mathbf{p})$ is both strictly proper and BIBO stable, then the satisfaction of the multivariable Popov criterion (12) implies that the system (4) is absolutely input-output stable. This is in fact an extension of the result in [17] to the case of multiple decoupled nonlinearities. By virtue of using the convolution algebra, the result in Theorem 3.5 is valid for rational and non-rational transfer functions as long as Assumption 3 holds. Therefore, the result is applicable to the system (4) whose LTI subsytems consist of lumped- and/or distributed-parameter components.

Following the well-known positive real lemma ([23, 24]), the criterion (12) is known to be equivalent to a linear matrix inequality (LMI) for the case in which all elements of $\mathbf{G}_{uv}(s, \mathbf{p})$ are strictly proper rational transfer functions. As a consequence, the stability test for this case can be carried out efficiently in practice by available computational tools. Further, based on the LMI, we develop a useful inequality for stabilizing the system by numerical methods. In conjunction with the method of inequalities, such an inequality leads to a numerical procedure for stabilizing the nonlinear system. In the numerical example, we demonstrate how to utilize the inequality in determining a controller for which the control system with two nonlinearities is absolutely input-output stable.

Although the determination of a numerical solution of inequality (24) in the space of design parameter vector $\mathbf{p}$ is in general a non-convex problem, an advantage of this approach is that, as long as a solution of the inequality exists, designers can specify any suitable (or implementable) controller structure.

One can see from Proposition 4.2 that inequality (24) is valid only when all elements of $\mathbf{G}_{uv}(s, \mathbf{p})$ are rational transfer functions. Regarding this limitation, it is interesting to derive an equivalent condition for the satisfaction of the criterion (12) that is suitable for numerical stabilization for the case in which the elements of $\mathbf{G}_{uv}(s, \mathbf{p})$ are non-rational transfer functions such as transfer functions of delay differential systems.

Acknowledgement

T. Chuman gratefully acknowledges the support from the 100th Anniversary Chulalongkorn University Fund for Doctoral Scholarship and the 90th Anniversary Chulalongkorn University Fund (Ratchadaphiseksomphot Endowment Fund). Both authors would like to thank an anonymous reviewer whose comments help us improve the manuscript.

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Appendix A. Proof of Lemmas 3.3 and 3.4

Proof of Lemma 3.3. Define

$$I = \sum_{i=1}^{n} \int_{0}^{\infty} y_i(t) x_i(t) dt.$$

By applying Parseval's theorem to equation (8), we have

$$\begin{aligned} I \;&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big[\mathbf{X}^*(j\omega) \mathbf{H}^*(j\omega) \mathbf{X}(j\omega) \\ &\quad + \mathbf{W}^*(j\omega) \mathbf{X}(j\omega) \Big] d\omega. \end{aligned} \tag{36}$$

Since $I \in \mathbb{R}$, it follow from (36) that

$$\begin{aligned} I \;&= \frac{1}{4\pi} \int_{-\infty}^{\infty} \Big[\mathbf{X}^*(j\omega) \big(\mathbf{H}(j\omega) + \mathbf{H}^*(j\omega) \big) \mathbf{X}(j\omega) \\ &\quad + \mathbf{W}^*(j\omega) \mathbf{X}(j\omega) + \mathbf{W}(j\omega) \mathbf{X}^*(j\omega) \Big] d\omega. \end{aligned} \tag{37}$$

By using condition (9), it follows from (37) that

$$\begin{aligned} I &\geq \frac{1}{4\pi}\int_0^\infty \Big[2\delta \mathbf{X}^*(j\omega)\mathbf{X}(j\omega) \\ &\quad +\mathbf{W}^*(j\omega)\mathbf{X}(j\omega)+\mathbf{W}(j\omega)\mathbf{X}^*(j\omega)\Big]d\omega \\ &= \frac{1}{4\pi}\sum_{i=1}^n\int_{-\infty}^\infty \left|\sqrt{2\delta}X_i(j\omega)+\frac{W_i(j\omega)}{\sqrt{2\delta}}\right|^2 d\omega \\ &\quad -\frac{1}{8\pi\delta}\sum_{i=1}^n\int_{-\infty}^\infty |W_i(j\omega)|^2 d\omega. \end{aligned} \tag{38}$$

Since the first integral of (38) is nonnegative, we have

$$\begin{aligned} I &\geq -\frac{1}{8\pi\delta}\sum_{i=1}^n\int_{-\infty}^\infty |W_i(j\omega)|^2 d\omega \\ &= -\frac{1}{4\delta}\sum_{i=1}^n\int_0^\infty |w_i(t)|^2 dt. \end{aligned}$$

Hence, the proof is complete. □

The following notation will be used in the proof of Lemma 3.4. For a function $x:\mathbb{R}_+\to\mathbb{R}$ and for any $T>0$, define the truncated function x_T as follows:

$$x_T(t) \triangleq \begin{cases} x(t), & 0\leq t\leq T \\ 0, & t>T \end{cases}$$

Proof of Lemma 3.4. From (4) and (11), it follows that for each i,

$$u_i(t) = r_i(t)+\sum_{j=1}^n\int_0^t g_{u_iv_j}(t-\tau)v_j(\tau)d\tau. \tag{39}$$

Differentiating both sides of (39) yields

$$\begin{aligned} &\dot u_i(t) = \dot r_i(t) \\ &+\sum_{j=1}^n\left[\int_0^t \dot g_{u_iv_j}(t-\tau)v_j(\tau)d\tau + g_{u_iv_j}(0)v_j(t)\right]. \end{aligned} \tag{40}$$

Let $T>0$ be fixed and let $r_{i,T}, \dot r_{i,T}, u_{i,T}$, and $v_{i,T}$ denote the truncated functions of $r_i, \dot r_i, u_i$, and v_i, respectively. Then, for any $q_i\in\mathbb{R}$ ($i=1,2,\ldots,n$), it follows from (39) and (40) that for each i,

$$\begin{aligned} &u_{i,T}(t)+q_i\dot u_{i,T}(t) = \\ &\quad r_{i,T}(t)+q_i\dot r_{i,T}(t)+q_i\sum_{j=1}^n g_{u_iv_j}(0)v_j(t) \\ &+\sum_{j=1}^n\int_0^t [g_{u_iv_j}(t-\tau)+q_i\dot g_{u_iv_j}(t-\tau)]v_{j,T}(\tau)d\tau. \end{aligned} \tag{41}$$

Let $\sigma\in(0,\beta)$. By subtracting $(1/k_i-\sigma)v_{i,T}(t)$ to both sides of (41) and then multiplying by $-e^{\alpha t}$ with sufficiently small $\alpha>0$, it follows that for each i,

$$\begin{aligned} y_i(t) &= w_i(t)-\sum_{j=1}^n q_i g_{u_iv_j}(0)e^{\alpha t}v_{j,T}(t) \\ &\quad -\sum_{j=1}^n\int_0^t e^{\alpha(t-\tau)}\big[g_{u_iv_j}(t-\tau) \\ &\quad +q_i\dot g_{u_iv_j}(t-\tau)\big]e^{\alpha\tau}v_{j,T}(\tau)d\tau \\ &\quad +\left(\frac{1}{k_i}-\sigma\right)e^{\alpha t}v_{i,T}(t) \end{aligned} \tag{42}$$

where $w_i(t)\triangleq -\left[r_{i,T}(t)+q_i\dot r_{i,T}(t)\right]e^{\alpha t}$ and

$$y_i(t)\triangleq\left[-u_{i,T}(t)-q_i\dot u_{i,T}(t)+\left(\frac{1}{k_i}-\sigma\right)v_{i,T}(t)\right]e^{\alpha t}.$$

All the terms in (42) belong to $\mathbb{L}^2$ due to the truncation at T. Then the Fourier transform of (42) is given by

$$\begin{aligned} &Y_i(j\omega) = W_i(j\omega)+\left(\frac{1}{k_i}-\sigma\right)V_{i,T}(j\omega-\alpha) \\ &-\sum_{j=1}^n\big[1+q_i(j\omega-\alpha)\big]G_{u_iv_j}(j\omega-\alpha)V_{j,T}(j\omega-\alpha) \end{aligned} \tag{43}$$

where $Y_i(j\omega), W_i(j\omega)$ and $V_i(j\omega)$ are the Fourier transforms of y_i, w_i and v_i, respectively. It can be verified that condition (12) implies that for each i, equation (43) satisfies conditions (8) and (9) with $\delta=\beta-\sigma$. By applying Lemma 3.3, we have

$$-\sum_{i=1}^n\int_0^\infty y_i(t)v_i(t)dt \leq \frac{1}{4\delta}\sum_{i=1}^n\int_0^\infty w_i^2(t)dt. \tag{44}$$

Let J_i denote the i^{th} term of the left-hand side of (44). Then it follows that for each i,

$$\begin{aligned} J_i &= \int_0^T\left(u_i-\frac{v_i}{k_i}\right)v_ie^{2\alpha t}dt \\ &\quad +q_i\int_0^T \dot u_iv_ie^{2\alpha t}dt+\sigma\int_0^T v_i^2e^{2\alpha t}dt \\ &= \int_0^T\left(u_i-\frac{\psi_i(u_i)}{k_i}\right)\psi_i(u_i)e^{2\alpha t}dt \\ &\quad -2q_i\alpha\int_0^T e^{2\alpha t}\left[\int_0^{u_i(t)}\psi_i(u_i)du_i\right]dt \\ &\quad +q_ie^{2\alpha T}\int_0^{u_i(T)}\psi_i(u_i)du_i - q_i\int_0^{u_i(0)}\psi_i(u_i)du_i \\ &\quad +\sigma\int_0^T v_i^2e^{2\alpha t}dt. \end{aligned} \tag{45}$$

Since $\psi_i(u_i)\in\text{sector}[\varepsilon,k_i-\varepsilon]$ where $\varepsilon>0$ is arbitrarily small, we have

$$\int_0^{u_i(t)}\psi_i(u_i)du_i\leq\frac{k_i}{2}u_i^2(t)$$

and

$$\left[u_i - \frac{\psi_i(u_i(t))}{k_i}\right] \psi_i(u_i(t)) \geq \frac{\varepsilon^2}{k_i} u_i^2(t).$$

Then it follows from (45) that for each i,

$$\begin{aligned} \frac{J_i}{\sigma} &\geq \frac{1}{\sigma} \int_0^T \left(\frac{\varepsilon^2}{k_i} - k_i q_i \alpha\right) e^{2\alpha t} u_i^2(t) dt \\ &+ \int_0^T e^{2\alpha t} v_i^2(t) dt - \frac{q_i}{\sigma} \int_0^{u_i(0)} \psi_i(u_i) du_i. \end{aligned} \tag{46}$$

Since $\varepsilon > 0$, since $q_i < \infty$, and since $k_i < \infty$, there always exists $\alpha > 0$ such that $(\varepsilon^2/k_i - k_i q_i \alpha) \geq 0$. Hence, inequalities (46) become

$$\frac{J_i}{\sigma} \geq \int_0^T e^{2\alpha t} v_i^2(t) dt - \frac{q_i}{\sigma} \int_0^{u_i(0)} \psi_i(u_i) du_i, \ \forall T > 0. \tag{47}$$

From (43), we have

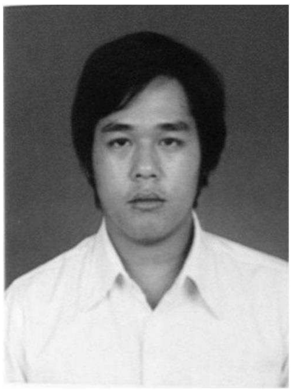
$$\sum_{i=1}^n \frac{J_i}{\sigma} \leq \frac{1}{4\sigma(\beta - \sigma)} \sum_{i=1}^n \int_0^T (r_i + q_i \dot{r}_i)^2 e^{2\alpha t} dt, \ \forall T > 0. \tag{48}$$

One can verify that $\sigma = \beta/2$ minimizes the right-hand side of (48). By substituting σ with $\beta/2$, it follows from (47) and (48) that

$$\begin{aligned} &\sum_{j=1}^n \int_0^T e^{2\alpha t} v_j^2(t) dt \leq \\ &\quad \frac{1}{\beta^2} \sum_{j=1}^n \int_0^T e^{2\alpha \tau} [r_j(\tau) + q_j \dot{r}_j(\tau)]^2 d\tau \\ &\quad + \sum_{j=1}^n \frac{2q_j}{\beta} \int_0^{u_j(0)} \psi_j(u_j) du_j, \ \forall T > 0. \end{aligned}$$

Hence, the lemma is proved. □

Suchin Arunsawatwong received his B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand, in 1985 and 1988, respectively, and Ph.D. degree in control engineering from the Control Systems Centre, University of Manchester Institute of Science and Technology, U.K., in 1995. He is currently an assistant professor at the Department of Electrical Engineering, Chulalongkorn University. His main research interests include control systems design by the method of inequalities and the principle of matching, delay differential systems, fractional-order control systems, control systems with nonlinearities, and power systems control applications.



Tadchanon Chuman received his B.Eng. and M.Eng. degrees in electrical engineering from Chulalongkorn University, Thailand in 2012 and 2014, respectively. He is currently studying for a Ph.D. at Control Systems Research Laboratory, the Department of Electrical Engineering, Chulalongkorn University. His main research interests include control systems design using the method of inequalities and the principle of matching, control systems with nonlinearities, and load frequency control.