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Article

Note on Fourier Transform of Hidden Variable Fractal Interpolation

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Abstract. This paper investigates the Fourier transform of a hidden variable fractal interpolation function with function scaling factors, which generalizes the Fourier transform of hidden variable fractal interpolation function with constant scaling factors. Furthermore, the Fourier transform of quadratic hidden variable fractal interpolation function with function scaling factors is also investigated. With an aim of maximizing the flexibility of hidden variable fractal interpolation function and quadratic hidden variable fractal interpolation function, a class of iterated function systems involving function scalings is chosen for the present study.

Keywords: Fourier transform, function scaling factors, hidden variable fractal interpolation function.

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1. Introduction

In the 1970s, the mathematician Mandelbrot explored a new geometry of nature that accepts the irregular structures of items including coastlines, lightning bolts, clouds, and molecular trajectories. Then, mathematicians had started studying in the late 19th century. The key characteristic of these objects, which Mandelbrot named *fractals*, is that their borders are so irregular that is not easy to comprehend how simple metric concepts and operations can be applied to them[1].

Conventional approaches to approximating data samples from scientific and natural phenomena are impacted by Euclidean geometry, where fundamental functions like linear, polynomial, trigonometric, and exponential functions play a significant role. Interpolation is a useful technique in approximation theory, which deals with the creation of a continuous function that abides by the sample data points. The classical techniques for interpolation typically produce smooth interpolants that are occasionally infinitely differentiable. As a result, the traditional method is insufficient for approximating non-differentiable functions and interpolating irregular data. In order to address this situation, Barnsley [2, 3] developed fractal interpolation, a new interpolation technique, based on the theory of iterated function systems. Their graphs serve as attractors for a class of hyperbolic iterated function systems (IFSs) composed of contractive maps that interpolate a given set of data. Fractal interpolation methods have been widely used in the natural sciences and engineering over the last 30 years (see, for example, refs. [4]-[14]) and have evolved into powerful tools for fitting and approximating many complex objects and patterns.

Barnsley et al.[3],[15] and Massopust[16] coined the word "hidden variable"and introduced the construction of hidden variable FIFs by using projections of the graphs of higher-dimensional FIFs to broaden the potential utility of FIFs and enhance the flexibility of interpolation. The benefits of HVFIFs are not only limited to flexibility but also provide greater diversity because their values are constantly dependent on hidden variables. For the same set of interpolation data, an HVFIF is more varied, exciting, and irregular than a FIF because the variables of a hidden variable FIF continuously differ on all of its IFS parameters. Moreover, by using the HVFIFs, self-affine and non-self-affine functions can both be interpreted concurrently. Later, Chand and Kapoor [17] briefly discussed the stability of a class of affine HVFIFs and described that any slight modification in the interpolation data leads to a small perturbation to the corresponding affine HVFIFs. Because HVFIFs are primarily generated by IFSs, it is also important to consider how the associated HVFIFs will change if the IFSs generating the HVFIFs undergo a small perturbation. The reader can look up more information about HVFIF in [18]-[31].

The Fourier transform is a mathematical transform that decomposes a function into its constituent sinusoidal components. It is one of the most important tools in signal processing and image processing, and it has many other applications in physics, engineering, and mathematics. The Fourier transform is a linear operation, which means that the Fourier transform of a sum of two functions is the sum of the Fourier transforms of the individual functions. This property is very useful for signal processing and image processing, as it allows us to decompose complex signals and images into simpler components. The Fourier transform also has a number of other useful properties, such as convolution, correlation, and symmetry. These properties make the Fourier transform a very powerful tool for analyzing signals and images.

In [32, 33], Alireza, et al. review fractal calculus and the analogues of both local Fourier transform with its related properties and Fourier convolution theorem are proposed in fractal calculus. The Fourier transform and fractal calculus are both valuable tools for studying fractals. However, they have different strengths and weaknesses. The Fourier transform is better suited for analyzing the frequency content of fractal objects, while fractal calculus is better suited for studying the dynamics of fractal systems.

In [34], Pan has discussed the Fourier series of fractal interpolation functions and demonstrated that complex fractal interpolation functions can be represented by the Fourier sine series and Fourier cosine series explicitly. Even though the Fourier series over the fractal functions provides an explicit structure, it can only be studied for the periodic functions. On the other hand, the Fourier transform can be used to study both periodic and non-periodic functions. To overcome the limitation of periodic cases, we investigate the Fourier transform of fractal interpolation functions with function scaling factors. Meanwhile, Barnsley has discussed the Fourier transform of the linear FIF in [2]. In this sequel, Navascués has presented the Fourier transform of α-fractal functions in [35, 36]. In the aforementioned discussions [35] and [36], the authors have considered the FIFs with constant scaling factors. In [35], the Fourier transform of the α-fractal function has been investigated, and in [36], the Fourier transform of the linear FIF and its relation with the fractal dimension have been explored. In [37], the authors explored the Fourier transform of linear FIF, quadratic FIF, and α-FIF with function scaling factors. However, an extensive review of the pertinent literature reveals that the Fourier transform of hidden variable FIF and quadratic hidden variable FIF with function scaling factors have still not been explored. With this motivation, this paper explores the Fourier transform of the hidden variable FIF and the quadratic hidden variable FIF with function scaling factors.

The structure of this article is as follows: In Section 2, the construction of the fractal interpolation function has been presented. The concepts of hidden variable fractal interpolation functions with function scaling factors have been broadly discussed in Section 3. Section 4 addresses the Fourier transform of the hidden variable FIF and the quadratic hidden variable FIF. The results of the present work are concluded in Section 5.

2. Fractal Interpolation Function

Let $h_0 < h_1 < \cdots < h_N$ be a partition of the real compact interval $\mathcal{I} = [h_0, h_N]$. Consider the given set of data points $\{(h_\iota, y_\iota) \in [h_0, h_N] \times \mathbb{R} : m = 0, 1, \ldots, N\}$. For $\iota \in \{1, 2, \ldots, N\}$, set $[h_{\iota-1}, h_\iota] = [h_{\iota-1}, h_\iota]$ and let $\mathcal{L}_\iota : [h_0, h_N] \to [h_{\iota-1}, h_\iota]$, be contractive homeomorphisms such that

$$\begin{cases} \mathcal{L}_\iota(h_0) = h_{\iota-1}, \\ \mathcal{L}_\iota(h_N) = h_\iota, \end{cases} \tag{1}$$

$$| \mathcal{L}_\iota(a) - \mathcal{L}_\iota(b) | \leq r \mid a - b \mid \quad \forall\, a, b \in I, \tag{2}$$

for $0 \leq r < 1$. Let $-1 < \alpha_\iota < 1, \iota = 1, 2, \ldots, N$, and $\mathcal{M} := I \times [c, d]$, $\min \mathcal{M}_\iota \leq c \leq \max \mathcal{M}_\iota \leq d$. For each ι, consider the continuous mapping $\mathcal{F}_\iota : \mathcal{M} \to \mathbb{R}$ such that

$$\begin{cases} \mathcal{F}_\iota(h_0, h_0) = h_{\iota-1}, \\ \mathcal{F}_\iota(h_N, h_N) = h_\iota, \text{ and} \end{cases} \tag{3}$$

$$|\mathcal{F}_\iota(h, y) - \mathcal{F}_\iota(h, z)| \leq |\alpha_\iota||y - z|, \quad h \in [h_0, h_N],\ h, z \in [c, d]. \tag{4}$$

Define the functions $w_\iota(h, y) = (\mathcal{L}_\iota(h), \mathcal{F}_\iota(h, y))$, $\forall\, \iota = 1, 2, \ldots, N$, then $\{\mathcal{M}, w_\iota : \iota = 1, 2, \ldots, N\}$ forms an IFS.

Theorem 1 *The above mentioned IFS admits a unique attractor G. G is the continuous function's graph $g : [h_0, h_N] \to \mathbb{R}$ which obeys $g(h_\iota) = y_\iota$ for $\iota = 0, 1, \ldots, N$.*

The function g obtained in Theorem 1 is termed to as a Fractal Interpolation Function (FIF) corresponding with the iterated function system $\{(\mathcal{L}_\iota(h), \mathcal{F}_\iota(h, y))\}_{\iota=1}^N$. Now, let us discuss the functional equation related with the fractal interpolation function. Let $\mathcal{F}$ be the set of continuous functions $f : [h_0, h_N] \to [c, d]$ such that $f(h_0) = y_0; f(h_N) = y_N$. $\mathcal{F}$ is a complete metric space with respect to the uniform norm, $\|.\|_\infty = \max\{| f(h) | : h \in [h_0, h_N]\}$. The Read-Bajraktarević (RB) operator $\mathcal{T} : \mathcal{F} \to \mathcal{F}$ is defined by

$$(\mathcal{T}f)(h) = \mathcal{F}_\iota(\mathcal{L}_\iota^{-1}(h), f \circ \mathcal{L}_\iota^{-1}(h)), \quad \forall\, h \in [h_{\iota-1}, h_\iota],\ \iota = 1, 2, \ldots, N.$$

On the metric space $(\mathcal{F}, \|.\|_\infty)$, $\mathcal{T}$ is a contraction, that is

$$\|\mathcal{T}f - \mathcal{T}g\|_\infty \leq | \alpha |_\infty \|f - g\|_\infty, \tag{5}$$

where $| \alpha |_\infty = \max\{| \alpha_\iota | : \iota = 1, 2, \ldots, N\}$. Since $| \alpha |_\infty < 1$, the iterated function system theory states that $\mathcal{T}$ has an unique fixed point on $\mathcal{F}$, that is to say, there exist unique $g(= \mathcal{T}(g)) \in \mathcal{F}$. Further, g satisfies the following functional equation,

$$g(h) = \mathcal{F}_\iota(\mathcal{L}_\iota^{-1}(h), g \circ \mathcal{L}_\iota^{-1}(h)),\ h \in [h_{\iota-1}, h_\iota]. \tag{6}$$

The most widely discussed IFS concerning the fractal interpolation functions is of the form,

$$\begin{cases} \mathcal{L}_\iota(h) = a_\iota h + b_\iota, \\ \mathcal{F}_\iota(h, y) = \alpha_\iota y + q_\iota(h), \end{cases} \tag{7}$$

where,

$$a_\iota = \frac{h_\iota - h_{\iota-1}}{h_N - h_0}, \quad b_\iota = \frac{h_N h_{\iota-1} - h_0 h_\iota}{h_N - h_0} \tag{8}$$

here α_ι is known to as vertical scaling factors of the transformation w_ι and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector.

Instead of constant scaling factors, one may take function scaling factors to define iterated function system for more flexibility. In [38], Wang and Yu introduced the construction of FIF using function scaling. The associated FIF satisfies

$$\mathcal{L}_\iota(h) = a_\iota h + b_\iota,\ \mathcal{F}_\iota(h, y) = \alpha_\iota(h) y + q_\iota(h), \tag{9}$$

where $\alpha_\iota(h)$ is the Lipschitz function defined on $\mathcal{I}$ satisfying $||\alpha_\iota||_\infty = \sup_{h \in I} |\alpha_\iota(h)| < 1$. According to theorem (1), the functional equation of new FIF $f : \mathcal{I} \to \mathbb{F}$ with function scaling associated with the IFS in (9) will be

$$f(h) = \alpha_\iota(\mathcal{L}_\iota^{-1}(h)) f(\mathcal{L}_\iota^{-1}(h)) + q_\iota(\mathcal{L}_\iota^{-1}(h)), \quad \text{for all } t \in \mathcal{I},\ \iota = 1, 2, \ldots, N. \tag{10}$$

The continuous function q_ι influences the shape of the FIF hence, in literature, the FIFs are classified as linear FIF, quadratic FIF and cubic FIF based on the choice of q_m. For instance, in [25, 26, 28], the authors refer to the *quadratic fractal interpolation function (QFIF)* as functions constructed using functions $\mathcal{F}_\iota$ of the form

$$\mathcal{F}_\iota(h,y) = \alpha_\iota(\mathcal{L}_\iota^{-1}(h)) f(\mathcal{L}_\iota^{-1}(h)) + h^2 + c_\iota h + d_\iota \quad \text{for each } \iota = 1,2,\ldots,N.$$

3. Hidden Variable Fractal Interpolation Function

The concepts of hidden variable fractal interpolation function with function scaling factors is broadly discussed below:

In order to define a hidden variable fractal interpolation function for the generalized data set $\{(h_\iota, y_\iota, z_\iota) \in [h_0, h_N] \times \mathbb{R}^2 : \iota = 0,1,\ldots,N\}$, where $\{z_\iota : \iota = 0,1,\ldots,N\}$ are real numbers, choose $\mathcal{L}_\iota(h)$ as in Eq.(1) and

$$\mathcal{F}_\iota(h,y,z) = A_\iota(y,z)^T + \big(p_\iota(h), q_\iota(h)\big)^T$$

where A_ι are upper-triangular matrices $\begin{bmatrix}\alpha_\iota(h) & \beta_\iota(h)\\ 0 & \gamma_\iota(h)\end{bmatrix}$ and $p_\iota(h)$, $q_\iota(h)$ are the real valued continuous functions for all $j \in J$. Note that the free variables and the constrained free variables are chosen as function scaling factors. For the construction of HVFIF, the generalized IFS on $\mathbb{R}^3$ is defined by

$$\{[h_0, h_N] \times \mathbb{R}^2; \mathcal{F}_\iota : j \in J\} \tag{11}$$

where $\mathcal{F}_\iota(h,y,z) = (\mathcal{L}_\iota(h), \mathcal{F}_\iota(h,y,z))$, $j \in J$. Then, the IFS (11) has unique attractor G^* which is the graph of the vector-valued function $\mathbf{f} = (f_1, f_2)$. Define the operator T' on the space of continuous functions from $[h_0, h_N]$ to $\mathbb{R}^2$ by

$$(\mathrm{T}'\mathbf{g})(h) = \mathcal{F}_\iota\big(\mathcal{L}_\iota^{-1}(h), \mathbf{g}\circ\big(\mathcal{L}_\iota^{-1}(h)\big)\big), \text{ for } h \in [h_{\iota-1}, h_\iota], j \in J.$$

By the Banach contraction principle, the contraction operator T' has a unique fixed point $\mathbf{f}$ which satisfies the functional equation,

$$(\mathrm{T}'\mathbf{f})(h) = \mathcal{F}_\iota\big(\mathcal{L}_\iota^{-1}(h), \mathbf{f}\circ\big(\mathcal{L}_\iota^{-1}(h)\big)\big), \text{ for } h \in [h_{\iota-1}, h_\iota],\ j \in J.$$

Then, the two components f_1 and f_2 of the vector-valued function $\mathbf{f}$ satisfies,

$$\begin{aligned} f_1\big(\mathcal{L}_\iota(h)\big) &= T_1' f_1(\mathcal{L}_\iota(h)) = F_{1m}(h, f_1(h), f_2(h)) \\ &= \alpha_\iota f_1(h) + \beta_\iota f_2(h) + p_\iota(h), \\ f_2\big(\mathcal{L}_\iota(h)\big) &= T_2' f_2(\mathcal{L}_\iota(h)) = F_{2m}(h, f_2(h)) \\ &= \gamma_\iota f_2(h) + q_\iota(h),\ h \in I. \end{aligned} \tag{12}$$

where $T_1' f_1$ and $T_2' f_2$ are the component wise operators of T'.

4. Fourier Transform of Hidden Variable Fractal Interpolation Functions

This section investigates the Fourier transform of linear hidden variable and quadratic hidden variable fractal interpolation function with function scaling factors. The Fourier transform of a continuous function $\mathbf{f} = (f_1, f_2)$ is given by

$$\hat{f}_1(w) = \int_0^1 e^{2\pi i w \tilde{h}} f_1(\tilde{h}) d\tilde{h}, \quad \hat{f}_2(w) = \int_0^1 e^{2\pi i w \tilde{h}} f_2(\tilde{h}) d\tilde{h}.$$

Irrespective of the periodicity, the Fourier transform can be defined for any continuous function. In this respect, the Fourier transform of a hidden variable FIF with function scaling factors is investigated in the following theorem.

Theorem 2 *Let* $\mathbf{f}(h) = (f_1, f_2)$ *be the linear hidden variable fractal interpolation function with function scaling factors determined by the IFS* $\{\mathcal{L}_\iota(h), \mathcal{F}_\iota(h,y,z)\}_{\iota=1}^N$. *If* $\|A_\iota'\| < 1$, *where* $A_\iota' = a_\iota \begin{bmatrix}\alpha_\iota(h) & \beta_\iota(h)\\ 0 & \gamma_\iota(h)\end{bmatrix}$, $||A_\iota'|| = max\{a_\iota||\alpha_\iota(h)|| + a_\iota||\beta_\iota(h)||, a_\iota||\gamma_\iota(h)||\}$, *where* $\alpha_\iota'(h) = a_\iota\alpha_\iota(h), \beta_\iota'(h) = a_\iota\beta_\iota(h), \gamma_\iota'(h) = a_\iota\gamma_\iota(h)$. *Further* $\hat{f}_1(w)$ *and* $\hat{f}_2(w)$ *are generated by the IFSs* $\{\mathcal{L}_\iota(h), \hat{F}_{1\iota}(h,\hat{y},\hat{z})\}_{\iota=1}^N$ *and* $\{\mathcal{L}_\iota(h), \hat{F}_{2\iota}(h,\hat{z})\}_{\iota=1}^N$ *respectively, where* $\hat{F}_{1\iota}(h,\hat{y},\hat{z}) = a_\iota\alpha_\iota(h)\hat{f}_1(h) + a_\iota\beta_\iota(h)\hat{f}_2(h) + \hat{p}_\iota(h)$, $\hat{F}_{2\iota}(h,\hat{z}) = a_\iota\gamma_\iota(h)\hat{f}_2(h) + \hat{q}_\iota(h)$. *Then, the Fourier transform of* f *is given by*

$$\begin{aligned} \hat{f}_1(w) =& \hat{P}(w) + \frac{1}{N} s_1(w) \hat{f}_1\left(\frac{w}{N}\right) - \sum_{\iota=1}^N e^{\frac{2\pi i w b_\iota}{N}} \\ & \int_0^1 \left[\alpha_\iota'(h) \int_0^h f_1(u) e^{\frac{2\pi i w u}{N}} du\right] dh \\ & + \frac{1}{N} s_2(w) \hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^N e^{\frac{2\pi i w b_\iota}{N}} \\ & \int_0^1 \left[\beta_\iota'(h) \int_0^h f_2(u) e^{\frac{2\pi i w u}{N}} du\right] dh, \\ \hat{f}_2(w) =& \hat{Q}(w) + \frac{1}{N} s(w) \hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^N e^{\frac{2\pi i w b_\iota}{N}} \\ & \int_0^1 \left[\gamma_\iota'(h) \int_0^h f_2(u) e^{\frac{2\pi i w u}{N}} du\right] dh, \end{aligned}$$

where

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},$$

$$\hat{Q}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{\mathcal{F}_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},$$

$$s(w) = \sum_{\iota=1}^{N}\gamma_\iota(1)e^{2\pi i w b_\iota}, s_1(w) = \sum_{\iota=1}^{N}\alpha_\iota(1)e^{2\pi i w b_\iota},$$

$$s_2(w) = \sum_{\iota=1}^{N}\beta_\iota(1)e^{2\pi i w b_\iota}.$$

Proof 2 *The Fourier transform of linear hidden variable FIF is defined by*

$$\hat{f}_1(w) = \int_0^1 e^{2\pi i w\tilde{h}} f_1(\tilde{h})d\tilde{h}, \quad \hat{f}_2(w) = \int_0^1 e^{2\pi i w\tilde{h}} f_2(\tilde{h})d\tilde{h}.$$

By using the functional equation

$$f_1(\tilde{h}) = \alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h})) + \beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h})) + p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})),$$

$$\begin{aligned}\hat{f}_1(w) = &\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} \\ &+ \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} \\ &+ \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}\end{aligned}$$

$$\hat{f}_1(w) = \hat{P}(w) + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})) f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})) f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}. \tag{13}$$

Consider the first term of equation (13),

$$\hat{P}(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}.$$

Changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\hat{P}(w) = a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{c_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}\left(1-\frac{1}{2\pi i w a_\iota}\right) + \frac{1}{2\pi i w a_\iota}\right] + \frac{d_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}-1\right]\right\}.$$

Suppose the equation (1) is considered with the interval $I = [0, 1]$ of uniform partition, $a_\iota = 1/N$, then

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}. \tag{14}$$

In the second term of equation (13), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\begin{aligned}&\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} \\ &= a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\alpha_\iota(1)\int_0^h f_1(h)e^{2\pi i w a_\iota u}dh \\ &- a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\alpha_\iota'(h)\int_0^h f_1(u)e^{2\pi i a_\iota u}du\right]dh.\end{aligned}$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\begin{aligned}&\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} \\ &= \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\alpha_\iota(1)\hat{f}_1\left(\frac{w}{N}\right) \\ &- \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\alpha_\iota'(h)\int_0^h f_1(u)e^{\frac{2\pi i w u}{N}}du\right]dh.\end{aligned}$$

In the third term of equation (13), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\begin{aligned}&\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} \\ &= a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\beta_\iota(1)\int_0^h f_2(h)e^{2\pi i w a_\iota u}dh \\ &- a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\beta_\iota'(h)\int_0^h f_2(u)e^{2\pi i a_\iota u}du\right]dh.\end{aligned}$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\beta_\iota(1)\hat{f}_2\left(\frac{w}{N}\right) - \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\beta_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh.$$

Therefore, from the equation (13),

$$\hat{f}_1(w) = \hat{P}(w) + \frac{1}{N}s_1(w)\hat{f}_1\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\alpha_\iota'(h)\int_0^h f_1(u)e^{\frac{2\pi i w u}{N}}du\right]dt + \frac{1}{N}s_2(w)\hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\beta_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh, \quad (15)$$

where

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}.$$

$$s_1(w) = \sum_{\iota=1}^{N}\alpha_\iota(1)e^{2\pi i w b_\iota}, \quad s_2(w) = \sum_{\iota=1}^{N}\beta_\iota(1)e^{2\pi i w b_\iota}. \quad (16)$$

Since f_2 satisfies the functional equation

$$f_2(h) = \gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h})) + q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})),$$

$$\hat{f}_2(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$

$$\hat{f}_2(w) = \hat{Q}(w) + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})) f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}. \quad (17)$$

Consider the first term of equation (17),

$$\hat{Q}(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}.$$

Changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\hat{Q}(w) = a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}\left(1-\frac{1}{2\pi i w a_\iota}\right) + \frac{1}{2\pi i w a_\iota}\right] + \frac{f_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}-1\right]\right\}.$$

Suppose the equation (1) is considered with the interval $I = [0,1]$ of uniform partition, $a_\iota = 1/N$, then

$$\hat{Q}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{f_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}. \quad (18)$$

In the second term of equation (17), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = t$, one has

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} = a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\gamma_\iota(1)\int_0^h f_2(h)e^{2\pi i w a_\iota u}dh - a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{2\pi i a_\iota u}du\right]dh.$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\gamma_\iota(1)\hat{f}_2\left(\frac{w}{N}\right) - \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dt.$$

Therefore, from the equation (17),

$$\hat{f}_2(w) = \hat{Q}(w) + \frac{1}{N}s(w)\hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dt, \quad (19)$$

where

$$\hat{Q}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right] + \frac{f_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},$$

$$s(w) = \sum_{\iota=1}^{N}\gamma_\iota(1)e^{2\pi i w b_\iota}. \quad (20)$$

Remark 1 *Theorem 2 discusses the Fourier transform of a linear hidden variable FIF with function scaling factors. As a particular case of Theorem 2, when the scalings are taken as constants the following results have been obtained,*

$$\hat{f}_1(w) = \hat{P}(w) + \frac{1}{N}s_1(w)\hat{f}_1\left(\frac{w}{N}\right) + \frac{1}{N}s_2(w)\hat{f}_2\left(\frac{w}{N}\right)$$
$$\hat{f}_2(w) = \hat{Q}(w) + \frac{1}{N}s(w)\hat{f}_2\left(\frac{w}{N}\right)$$

where

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right]\frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},$$

$$\hat{Q}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right) + \frac{1}{\frac{2\pi i w}{N}}\right]\frac{f_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},$$

$$s(w) = \sum_{\iota=1}^{N}\gamma_\iota e^{2\pi i w b_\iota}, \quad s_1(w) = \sum_{\iota=1}^{N}\alpha_\iota e^{2\pi i w b_\iota},$$
$$s_2(w) = \sum_{\iota=1}^{N}\beta_\iota e^{2\pi i w b_\iota}.$$

Example 1 *Consider a data set:* $\{(0,0,0), (1/5,1/4,1/2),(1/2,1/6,3/4),(1,1,1)\}$. *The scale vectors* $\alpha_3 = (0.3,0.5,0.6), \beta_3 = (0.4,0.6,0.8), \gamma_3 = (0.3,0.5,-0.5)$ *are taken to be a linear hidden variable FIF with constant scaling factors and the linear hidden variable FIF with function scaling factors are* $\alpha_3(h) = \left(\frac{\sqrt{h}}{4}, \frac{e^h}{4}, \frac{sin\pi h}{5}\right)$, $\beta_3(h) = \left(\frac{h-1}{3}, \frac{cos\pi h}{4}, \frac{h}{4}\right)$, $\gamma_3(h) = \left(\frac{sin\pi h}{2}, \frac{cosh}{10}, \frac{\sqrt{h+1}}{5}\right)$. *The coefficients involved in the corresponding IFSs are provided in Table 1. Figure 1(a) reveals the graphical representation of the linear hidden variable FIF with constant scaling factors of a non-self-affine fractal function* (f_1), *while Fig. 1(b) depicts the graphical representation of the linear hidden variable FIF with constant scaling factors of a self-affine FIF* (f_2).

Table 1. Estimated coefficients associated with the linear HVFIF.

			$\mathcal{L}_\iota(h)$		$\mathcal{F}_\iota(h,y)$			
α_ι	β_ι	γ_ι	a_ι	b_ι	c_ι	d_ι	e_ι	f_ι
0.3	0.4	0.3	0.2	0	-0.45	0	0.2	0
0.5	0.6	0.3	0.3	0.2	-1.18	0.25	-0.25	0.5
0.6	0.8	-0.5	0.5	0.5	-0.56	0.5	0.75	0.75

Definition 1 *In the definition of hidden variable fractal interpolation function, if the scaling factors are chosen as continuous functions and* $p_\iota(h)$ *and* $q_\iota(h)$ *are chosen as quadratic functions of the form,*

$$p_\iota(h) = h^2 + c_\iota h + d_\iota,$$
$$q_\iota(h) = h^2 + e_\iota h + \mathcal{F}_\iota, \ \forall \ \iota = 1,2,\ldots,N \tag{21}$$

then the fixed point equations,

$$f_1\big(\mathcal{L}_\iota(h)\big) = \alpha_\iota(h)f_1(h) + \beta_\iota(h)f_2(h) + h^2 + c_\iota h + d_\iota,$$
$$f_2\big(\mathcal{L}_\iota(h)\big) = \gamma_\iota(h)f_2(h) + h^2 + e_\iota h + \mathcal{F}_\iota, \ h \in I. \tag{22}$$

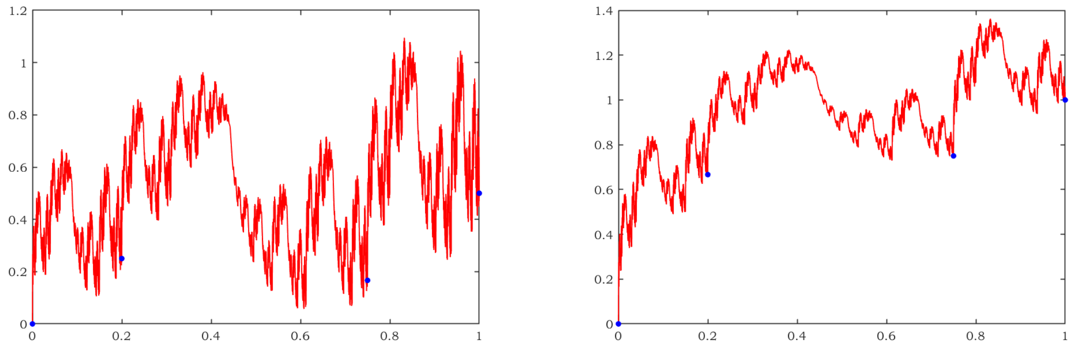
are called the quadratic hidden variable fractal interpolation function (QHVFIF) and quadratic fractal interpolation function (QFIF) with function scaling factors respectively.

The following theorem explore the Fourier transform of a quadratic hidden variable FIF with function scaling factors.

Theorem 3 *Let* $\mathbf{f}(h) = (f_1, f_2)$ *be the quadratic hidden variable fractal interpolation function with function scaling factors determined by the IFS* $\{\mathcal{L}_\iota(h), \mathcal{F}_\iota(h,y,z)\}_{\iota=1}^N$. *If* $\|A'_\iota\| < 1$, *where* $A'_\iota = a_\iota\begin{bmatrix}\alpha_\iota(h) & \beta_\iota(h)\\ 0 & \gamma_\iota(h)\end{bmatrix}$, $\|A'_\iota\| = max\{a_\iota\|\alpha_\iota(h)\| + a_\iota\|\beta_\iota(h)\|, a_\iota\|\gamma_\iota(h)\|\}$, *where* $\alpha'_\iota(h) = a_\iota\alpha_\iota(h), \beta'_\iota(h) = a_\iota\beta_\iota(h), \gamma'_\iota(h) = a_\iota\gamma_\iota(h)$. *Further* $\hat{f}_1(w)$ *and* $\hat{f}_2(w)$ *are generated by the IFSs* $\{\mathcal{L}_\iota(h), \hat{F}_{1\iota}(h,\hat{y},\hat{z})\}_{\iota=1}^N$ *and* $\{\mathcal{L}_\iota(h), \hat{F}_{2\iota}(h,\hat{z})\}_{\iota=1}^N$ *respectively, where* $\hat{F}_{1\iota}(h,\hat{y},\hat{z}) = a_\iota\alpha_\iota(h)\hat{f}_1(h) + a_\iota\beta_\iota(h)\hat{f}_2(h) + \hat{p}_\iota(h)$, $\hat{F}_{2\iota}(h,\hat{z}) = a_\iota\gamma_\iota(h)\hat{f}_2(h) + \hat{q}_\iota(h)$. *Then, the Fourier transform of* f *is given by*

$$\hat{f}_1(w) = \hat{P}(w) + \frac{1}{N}s_1(w)\hat{f}_1\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\alpha'_\iota(h)\int_0^h f_1(u)e^{\frac{2\pi i w u}{N}}du\right]dh + \frac{1}{N}s_2(w)\hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\beta'_\iota(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh,$$

$$\hat{f}_2(w) = \hat{Q}(w) + \frac{1}{N}s(w)\hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}} \int_0^1\left[\gamma'_\iota(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh,$$



(a) (b)

Fig. 1. Hidden variable fractal interpolation function:(a) Non-self-affine f_1, (b) self-affine f_2.

where

$$\hat{P}(w) =\frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi iwb_\iota}\left\{\left[\frac{e^{\frac{2\pi iw}{N}}}{\frac{2\pi iw}{N}}\right] - \frac{1}{\frac{\pi iw}{N}}\frac{1}{\frac{2\pi iw}{N}}\right.$$
$$\left[e^{\frac{2\pi iw}{N}}\left(1-\frac{1}{\frac{2\pi iw}{N}}\right)+\frac{1}{\frac{2\pi iw}{N}}\right]+\frac{c_\iota}{\frac{2\pi iw}{N}}\left[e^{\frac{2\pi iw}{N}}\right.$$
$$\left.\left.\left(1-\frac{1}{\frac{2\pi iw}{N}}\right)+\frac{1}{\frac{2\pi iw}{N}}\right]+\frac{d_\iota}{\frac{2\pi iw}{N}}\left[e^{\frac{2\pi iw}{N}}-1\right]\right\},$$

$$\hat{Q}(w) =\frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi iwb_\iota}\left\{\left[\frac{e^{\frac{2\pi iw}{N}}}{\frac{2\pi iw}{N}}\right] - \frac{1}{\frac{\pi iw}{N}}\frac{1}{\frac{2\pi iw}{N}}\right.$$
$$\left[e^{\frac{2\pi iw}{N}}\left(1-\frac{1}{\frac{2\pi iw}{N}}\right)+\frac{1}{\frac{2\pi iw}{N}}\right]+\frac{e_\iota}{\frac{2\pi iw}{N}}\left[e^{\frac{2\pi iw}{N}}\right.$$
$$\left.\left.\left(1-\frac{1}{\frac{2\pi iw}{N}}\right)+\frac{1}{\frac{2\pi iw}{N}}\right]+\frac{\mathcal{F}_\iota}{\frac{2\pi iw}{N}}\left[e^{\frac{2\pi iw}{N}}-1\right]\right\},$$

$$s_1(w) = \sum_{\iota=1}^{N}\alpha_\iota(1)e^{2\pi iwb_\iota}, \quad s_2(w) = \sum_{\iota=1}^{N}\beta_\iota(1)e^{2\pi iwb_\iota},$$
$$s(w) = \sum_{\iota=1}^{N}\gamma_\iota(1)e^{2\pi iwb_\iota}.$$

Proof 3 *The Fourier transform of quadratic hidden variable FIF is defined by*

$$\hat{f}_1(w) = \int_0^1 e^{2\pi iw\tilde{h}}f_1(\tilde{h})d\tilde{h}, \quad \hat{f}_2(w) = \int_0^1 e^{2\pi iw\tilde{h}}f_2(\tilde{h})d\tilde{h}.$$

By using the functional equation

$$f_1(\tilde{h}) =\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h})) + \beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))$$
$$f_2(\mathcal{L}_\iota^{-1}(\tilde{h})) + p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})),$$

$$\hat{f}(w) =\sum_{\iota=1}^{N}\int_0^1 e^{2\pi iw\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$+\sum_{\iota=1}^{N}\int_0^1 e^{2\pi iw\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$+\sum_{\iota=1}^{N}\int_0^1 e^{2\pi iw\tilde{h}}p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$

$$\hat{f}(w) =\hat{P}(w) + \sum_{\iota=1}^{N}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))\int_0^1 e^{2\pi iw\tilde{h}}f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$+\sum_{\iota=1}^{N}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))\int_0^1 e^{2\pi iw\tilde{h}}f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}. \tag{23}$$

Consider the first term of equation (23),

$$\hat{P}(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi iw\tilde{h}}p_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}.$$

Changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$*, one has*

$$\hat{P}(w) =a_\iota\sum_{\iota=1}^{N}e^{2\pi iwb_\iota}\int_0^1 e^{2\pi iwa_\iota t}p_\iota(h)dh$$
$$=a_\iota\sum_{\iota=1}^{N}e^{2\pi iwb_\iota}\left\{\left[\frac{e^{2\pi iwa_\iota}}{2\pi iwa_\iota}\right]-\frac{1}{\pi iwa_\iota}\right.$$
$$\left[\frac{e^{2\pi iwa_\iota}}{2\pi iwa_\iota}-\frac{e^{2\pi iwa_\iota}}{(2\pi iwa_\iota)^2}+\frac{1}{(2\pi iwa_\iota)^2}\right]$$
$$+\frac{c_\iota}{2\pi iwa_\iota}\left[e^{2\pi iwa_\iota}\left(1-\frac{1}{2\pi iwa_\iota}\right)\right.$$
$$\left.\left.+\frac{1}{2\pi iwa_\iota}\right]+\frac{d_\iota}{2\pi iwa_\iota}\left[e^{2\pi iwa_\iota}-1\right]\right\}$$

Suppose the equation (1) is considered with the interval $I = [0, 1]$ of uniform partition, $a_\iota = 1/N$, then

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\left[\frac{e^{\frac{2\pi i w}{N}}}{\frac{2\pi i w}{N}}\right] - \frac{1}{\frac{\pi i w}{N}}\frac{1}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right] + \frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right] + \frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}. \quad (24)$$

In the second term of equation (23), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = t$, one has

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$= a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\alpha_\iota(1)\int_0^h f_1(h)e^{2\pi i w a_\iota u}dh$$
$$- a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\alpha'_\iota(h)\int_0^h f_1(u)e^{2\pi i a_\iota u}du\right]dh.$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\alpha_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_1(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$= \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\alpha_\iota(1)\hat{f}_1\left(\frac{w}{N}\right)$$
$$- \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\alpha'_\iota(h)\int_0^h f_1(u)e^{\frac{2\pi i w u}{N}}du\right]dh.$$

In the third term of equation (23), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$= a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\beta_\iota(1)\int_0^h f_2(h)e^{2\pi i w a_\iota u}dh$$
$$- a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\beta'_\iota(h)\int_0^h f_2(u)e^{2\pi i a_\iota u}du\right]dh.$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\beta_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$
$$= \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\beta_\iota(1)\hat{f}_2\left(\frac{w}{N}\right)$$
$$- \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\beta'_\iota(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh.$$

Therefore, from the equation (23),

$$\hat{f}_1(w) = \hat{P}(w) + \frac{1}{N}s_1(w)\hat{f}_1\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}}\int_0^1\left[\alpha'_\iota(h)\int_0^h f_1(u)e^{\frac{2\pi i w u}{N}}du\right]dh + \frac{1}{N}s_2(w)\hat{f}_2\left(\frac{w}{N}\right) - \sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}}\int_0^1\left[\beta'_\iota(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dh,$$

where

$$\hat{P}(w) = \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\left[\frac{e^{\frac{2\pi i w}{N}}}{\frac{2\pi i w}{N}}\right] - \frac{1}{\frac{\pi i w}{N}}\frac{1}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right] + \frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right] + \frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}.$$

$$s_1(w) = \sum_{\iota=1}^{N}\alpha_\iota(1)e^{2\pi i w b_\iota}, \quad s_2(w) = \sum_{\iota=1}^{N}\beta_\iota(1)e^{2\pi i w b_\iota}.$$

Since f_2 satisfies the functional equation

$$f_2(h) = \gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h})) + q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h})),$$

$$\hat{f}_2(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h} + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}$$

$$\hat{f}_2(w) = \hat{Q}(w) + \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}. \quad (25)$$

Consider the first term of equation (25),

$$\hat{Q}(w) = \sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}q_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}.$$

Changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = h$, one has

$$\begin{aligned}\hat{Q}(w) =& a_\iota \sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}\right.\right.\\ &\left(1-\frac{1}{2\pi i w a_\iota}\right)+\left.\frac{1}{2\pi i w a_\iota}\right]\\ &\left.+\frac{\mathcal{F}_\iota}{2\pi i w a_\iota}\left[e^{2\pi i w a_\iota}-1\right]\right\}.\end{aligned}$$

Suppose the equation (1) is considered with the interval $I = [0, 1]$ of uniform partition, $a_\iota = 1/N$, then

$$\begin{aligned}\hat{Q}(w) =& \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)\right.\right.\\ &\left.\left.+\frac{1}{\frac{2\pi i w}{N}}\right]+\frac{\mathcal{F}_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\}. \qquad (26)\end{aligned}$$

In the second term of equation (25), changing the variable $\mathcal{L}_\iota^{-1}(\tilde{h}) = t$, one has

$$\begin{aligned}&\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}\\ &= a_\iota\sum_{\iota=1}^{N}\int_0^1 \gamma_\iota(h)f_2(h)e^{2\pi i w(\mathcal{L}_\iota(h))}dt\\ &= a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\gamma_\iota(1)\int_0^h f_2(h)e^{2\pi i w a_\iota u}dt\\ &- a_\iota\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{2\pi i a_\iota u}du\right]dt.\end{aligned}$$

Similarly for uniform partition, $a_\iota = 1/N$, thus

$$\begin{aligned}&\sum_{\iota=1}^{N}\int_0^1 e^{2\pi i w\tilde{h}}\gamma_\iota(\mathcal{L}_\iota^{-1}(\tilde{h}))f_2(\mathcal{L}_\iota^{-1}(\tilde{h}))d\tilde{h}\\ &= \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\gamma_\iota(1)\hat{f}_2\left(\frac{w}{N}\right)\\ &- \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dt.\end{aligned}$$

Therefore, from the equation (25),

$$\begin{aligned}\hat{f}_2(w) =& \hat{Q}(w)+\frac{1}{N}s(w)\hat{f}_2\left(\frac{w}{N}\right)-\sum_{\iota=1}^{N} e^{\frac{2\pi i w b_\iota}{N}}\\ &\int_0^1\left[\gamma_\iota'(h)\int_0^h f_2(u)e^{\frac{2\pi i w u}{N}}du\right]dt,\end{aligned}$$

where

$$\begin{aligned}\hat{Q}(w) =& \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)\right.\right.\\ &\left.\left.+\frac{1}{\frac{2\pi i w}{N}}\right]+\frac{\mathcal{F}_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},\\ s(w) =& \sum_{\iota=1}^{N}\gamma_\iota(1)e^{2\pi i w b_\iota}.\end{aligned}$$

Remark 2 *Theorem 3 discusses the Fourier transform of a quadratic hidden variable FIF with function scaling factors. When the scalings are assumed to be constants, the following results have been obtained as a special case of Theorem 3.*

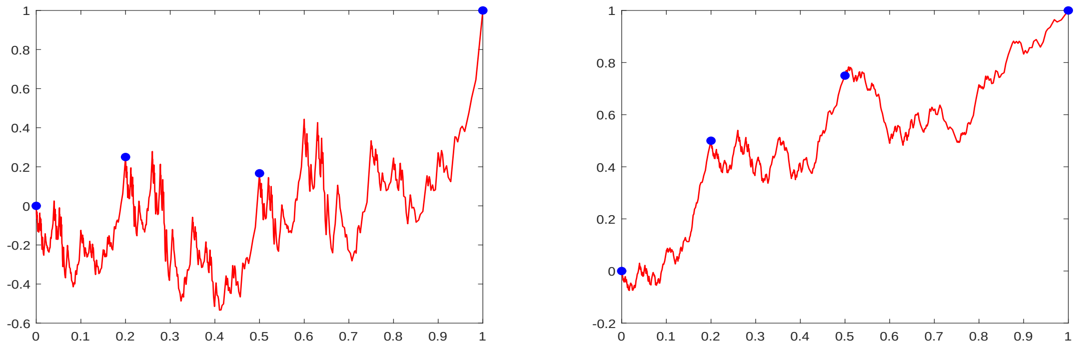
$$\hat{f}_1(w) = \hat{P}(w)+\frac{1}{N}s_1(w)\hat{f}_1\left(\frac{w}{N}\right)+\frac{1}{N}s_2(w)\hat{f}_2\left(\frac{w}{N}\right)$$

$$\hat{f}_2(w) = \hat{Q}(w)+\frac{1}{N}s(w)\hat{f}_2\left(\frac{w}{N}\right)$$

where

$$\begin{aligned}\hat{P}(w) =& \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\left[\frac{e^{\frac{2\pi i w}{N}}}{\frac{2\pi i w}{N}}\right]-\frac{1}{\frac{\pi i w}{N}}\frac{1}{\frac{2\pi i w}{N}}\right.\\ &\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right]+\frac{c_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\right.\\ &\left.\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right]+\left.\frac{d_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},\\ \hat{Q}(w) =& \frac{1}{N}\sum_{\iota=1}^{N} e^{2\pi i w b_\iota}\left\{\left[\frac{e^{\frac{2\pi i w}{N}}}{\frac{2\pi i w}{N}}\right]-\frac{1}{\frac{\pi i w}{N}}\frac{1}{\frac{2\pi i w}{N}}\right.\\ &\left[e^{\frac{2\pi i w}{N}}\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right]+\frac{e_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}\right.\\ &\left.\left(1-\frac{1}{\frac{2\pi i w}{N}}\right)+\frac{1}{\frac{2\pi i w}{N}}\right]+\left.\frac{\mathcal{F}_\iota}{\frac{2\pi i w}{N}}\left[e^{\frac{2\pi i w}{N}}-1\right]\right\},\\ s_1(w) =& \sum_{\iota=1}^{N}\alpha_\iota e^{2\pi i w b_\iota}, \quad s_2(w) = \sum_{\iota=1}^{N}\beta_\iota e^{2\pi i w b_\iota},\\ s(w) =& \sum_{\iota=1}^{N}\gamma_\iota e^{2\pi i w b_\iota}.\end{aligned}$$

Example 2 *Consider a data set: $\{(0,0,0), (1/5,1/4,1/2),(1/2,1/6,3/4),(1,1,1)\}$. The scale vectors $\alpha_3 = (0.3,0.5,0.6), \beta_3 = (0.4,0.6,0.8), \gamma_3 = (0.3,0.5,-0.5)$ are taken to be a quadratic hidden variable FIF with constant scaling factors and the quadratic hidden variable FIF with function scaling factors are $\alpha_3(h) = \left(\frac{\sqrt{h}}{4}, \frac{e^h}{4}, \frac{sin\pi t}{5}\right)$, $\beta_3(h) = \left(\frac{h-1}{3}, \frac{cos\pi h}{4}, \frac{t}{4}\right)$, $\gamma_3(h) = \left(\frac{sin\pi h}{2}, \frac{cosh}{10}, \frac{\sqrt{h+1}}{5}\right)$. Figure 2(a) displays the graphical representation of the*



(a) (b)

Fig. 2. Quadratic hidden variable fractal interpolation function:(a) Non-self-affine f_1, (b) self-affine f_2.

quadratic hidden variable FIF with constant scaling factors of a non-self-affine fractal function (f_1), while Fig. 2(b) depicts the graphical representation of the quadratic hidden variable FIF with constant scaling factors of a self-affine FIF (f_2).

Table 2. Estimated coefficients associated with the quadratic HVFIF.

			$\mathcal{L}_\iota(h)$		$\mathcal{F}_\iota(h,y)$			
α_ι	β_ι	γ_ι	a_ι	b_ι	c_ι	d_ι	e_ι	f_ι
0.3	0.4	0.3	0.2	0	-1.45	0	-0.8	0
0.5	0.6	0.3	0.3	0.2	-2.18	0.25	-1.25	0.5
0.6	0.8	-0.5	0.5	0.5	-1.56	0.16	-0.25	0.75

5. Conclusion

The Fourier transform of hidden variable FIF and the quadratic hidden variable FIF has been investigated using function scaling factors, which are a generalisation of constant scaling factors. The collection of scaling factors used as a function scaling factor in the current work makes the HVFIF more diverse and flexible, making it suitable for both regular and irregular interpolation data. HVFIF with function scaling factors provides superior outcomes for approximating more complex natural phenomena, such as wave functions, weather forecasts, share price fluctuations, etc., than HVFIF with constant scaling factors.

6. Compliance with Ethical Standards

Data availability statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of interest: The authors declare that they have no conflicts of interest.

Statements on ethical standards: This article does not contain any studies involving human participants and animals performed by any of the authors.

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